

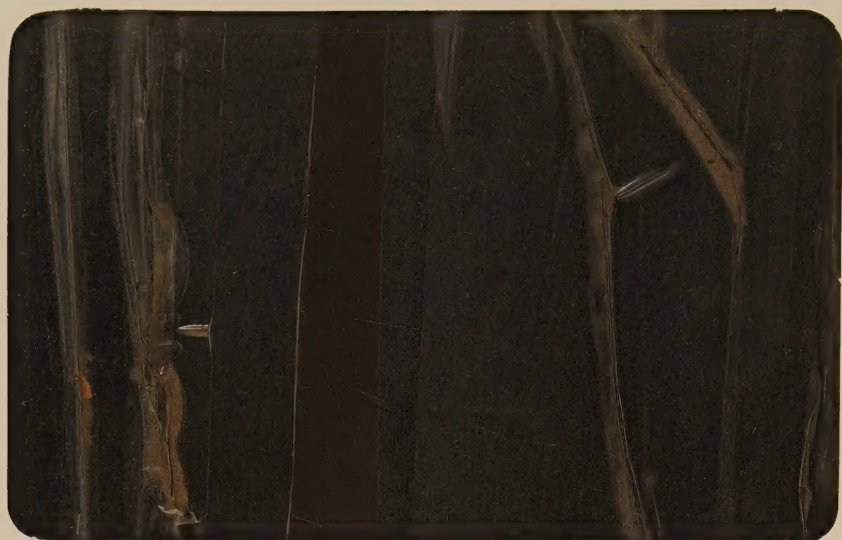


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FLOW AND PARTICLE BEHAVIOUR
IN FLOW THROUGH CHANNELS
WITH SUSPENDED PARTICLES

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DISSERTATION
JUNE 1974



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FLOW AND PARTICLE BEHAVIOUR IN FLOW THROUGH CHANNELS WITH SUSPENDED PARTICLES

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This thesis consists of this summary and the following papers:

- I. Johansson, H., Olgård, G. and Jernqvist, A.
Radial particle migration in plug flow - a method for
solid-liquid separation and fractionation
Chem.Eng.Sci. 25 365 1970
- II. Johansson, H. and Loyd, D.
On the flow near a liquid surface in a vertical channel
with moving walls
Chem.Eng.Sci. 27 157 1972
- III. Johansson, H.
A numerical solution of the flow around a circular
cylinder between two parallel plates
Department of Chemical Engineering, Lund Institute of
Technology, Lund, Report no 74-F-2, 1974
- IV. Johansson, H.
A numerical solution of the flow around a sphere in
a circular cylinder
Department of Chemical Engineering, Lund Institute of
Technology, Lund, Report no 74-F-3, 1974
(Submitted for publication in Chem.Eng.Comm.)

In the following these papers are referred to by their roman
numbers.

INTRODUCTION

The flow phenomena associated with suspended particles in a fluid that is flowing through a channel are of great importance in many branches of science and technology. The flow of fiber suspensions in paper making, latex particles in emulsion paints, red cells in blood, pipeline transportation of coal and ores and pneumatic conveyance of catalyst pellets are only a few examples in diverse fields.

The application of the fundamental equations of motion to the flow of multiparticle systems is complicated. Although the basic equations for the microscopic behaviour are well known, it is difficult to obtain even approximate solutions for simple systems. The macroscopic balances used in most practical applications often include empirical relationships to achieve utility.

In order to construct tractable mathematical models for the microscopic behaviour it is necessary to resort to a number of simplifications. A common condition when treating multiparticle systems is the assumption of laminar flow. For turbulent flow the mathematical analysis is very difficult even in the case of a single spherical particle.

If the flow is assumed to be laminar, Newtonian and incompressible, the governing equations of motion are the Navier-Stokes equation and the equation of continuity

$$\nabla p - \mu \nabla^2 \bar{\mathbf{v}} + \rho \bar{\mathbf{v}} \cdot \nabla \bar{\mathbf{v}} = 0$$

$$\nabla \cdot \bar{\mathbf{v}} = 0$$

For a multiparticle system with a finite number of particles the boundary conditions are as follows:

$$\bar{\mathbf{v}} = \bar{\mathbf{u}}_i + \bar{\boldsymbol{\omega}}_i \times \bar{\mathbf{r}}_i \quad \text{on } S_i$$

$$\bar{\mathbf{v}} = 0 \quad \text{on } S_w$$

$$\bar{\mathbf{v}} = \bar{\mathbf{U}} \quad \text{at infinity}$$

S_i is the surface of particle number i and S_w the surface of the boundary walls. $\bar{u}_i$ is the instantaneous translational velocity of an arbitrary point O_i affixed to the particle: $\bar{\omega}_i$ is the instantaneous angular velocity of the particle and $\bar{r}_i$ is the position vector of any point relative to an origin at O_i . $\bar{U}_i$ is an arbitrary velocity field that itself satisfies the equations of motion.

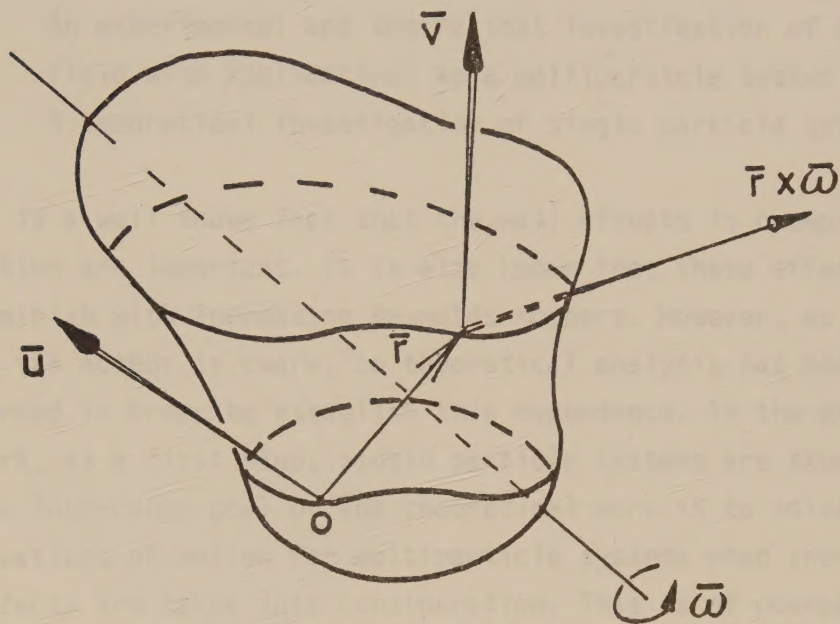


FIGURE 1. Definition sketch of the translational and rotational motion of a particle.

A complication is the non-linear fluid-inertia term in the Navier-Stokes equation. Analytical solutions are therefore restricted to zero Reynolds numbers, when the inertia term vanishes (Stokes flow or creeping motion), or to very low Reynolds numbers when it is possible to simplify the Navier-Stokes equation to a quasi-linear differential equation.

Still it is very difficult to obtain solutions for multiparticle systems. In the study of suspensions, where particles may assume arbitrary positions, it is not possible to obtain simple mathematical expressions. However, a generalized solution technique for creeping motion is outlined by Brenner [1] for multiparticle systems with the fluid at rest at infinity.

The calculation is a complicated matter and not tractable unless the particle systems are dilute.

This thesis deals with the flow and particle behaviour in flow through channels with suspended particles, when inertia effects are not negligible. Three kinds of systems are treated.

1. An experimental investigation of a multiparticle system
2. An experimental and theoretical investigation of a flow field with applications to a multiparticle system
3. A theoretical investigation of single particle systems

It is a well known fact that the wall effects in creeping motion are important. It is also known that these effects diminish with increasing Reynolds numbers. However, as far as the author is aware, no theoretical analysis has been performed in order to establish this dependence. In the present work, as a first step, single particle systems are studied. The long-range goal of the theoretical work is to solve the equations of motion for multiparticle systems when inertia effects are taken into consideration. This is of course an ideal goal which, if ever possible, requires great deal of future work.

SUMMARY

Papers I - II

When a liquid flows along a tube behind an advancing liquid-air meniscus, neutrally bouyant particles suspended in the liquid concentrate immediately behind the meniscus. This effect, the so-called meniscus effect, was first reported by Vejlens [2], who found that the concentration of red blood cells was increased behind the leading meniscus when blood was drawn through capillary tubes. Forgacs et al. [3] observed this behaviour with pulp-fiber suspensions, and Karnis and Mason [4] investigated the phenomena with suspensions of rigid spheres.

In paper I the use of this effect to separate and fractionate neutrally bouyant particles in a suspension is demonstrated. If the particle-liquid suspension flows in the form of a discrete plug, bounded by two liquid-air menisci, there is an increased concentration of particles behind the leading meniscus and a decreased concentration of particles at the trailing meniscus. By chopping off the plug at a suitable point, two streams are obtained: one with higher concentration than the original suspension and one with lower. This effect is exploited in the AJFO-fractionator, used in industrial applications (see fig 2).



The suspension is fed at the centre of the fractionator and on rotation plugs are formed in the helical duct. The plugs are impelled towards the periphery, during which separation and fractionation occurs. At the periphery the plugs drop into various compartments. By varying the arrangement of the compartments it is possible to obtain various fractions of the suspension.

FIGURE 2. Principle of the AJFO-fractionator.

The investigation was performed with a rather idealized suspension: polystyrene spheres in a water-glycerine solution with such a density that the spheres are neutrally bouyant. The influence of various parameters on the concentration profiles in the plugs was examined.

The following parameters were investigated: the particle diameter, the tube diameter, the plug velocity and the initial concentration of the suspension. The plug length and the distance covered by the plugs before they were examined were constant during the experiments.

It was found that an increased ratio of sphere diameter to tube diameter and increased plug velocity (applies to low velocities) gave favourable concentration profiles for separation. When the initial concentration was increased the concentration profiles became flatter. The increase of particle concentration at the leading meniscus is thus not so pronounced.

The dependence on the ratio sphere diameter/tube diameter indicates that the meniscus effect can be used for fractionation; see fig 3. This is indeed true, and furthermore the fractionation effect could be predicted, when the concentration profiles of the individual particle sizes are known.

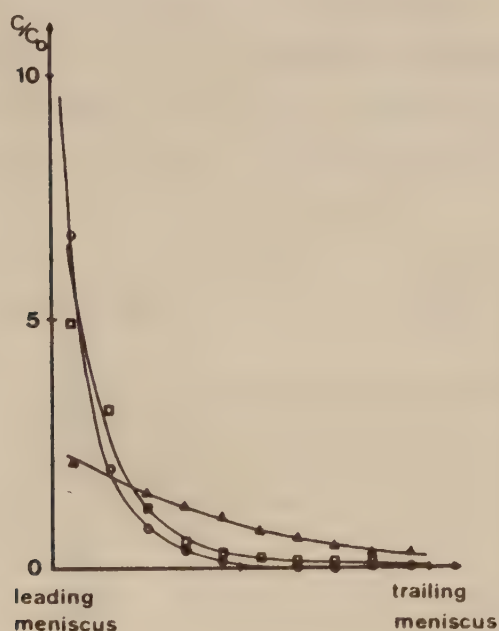


FIGURE 3. Concentration profiles for various ratios of sphere diameter to tube diameter.

- $(7.5 - 10) \cdot 10^{-2}$
- ◻ $(3.8 - 5.0) \cdot 10^{-2}$
- ◄ $(1.3 - 2.5) \cdot 10^{-2}$

Initial concentration $C_0 = 0.01 \text{ m}^3/\text{m}^3$

Plug velocity $U = 0.5 \text{ m/s}$

Tube diameter $D = 0.02 \text{ m}$

It was found that an increased ratio of sphere diameter to tube diameter and increased plug velocity (applies to low velocities) gave favourable concentration profiles for separation. When the initial concentration was increased, the concentration profile became flatter. The increase of particle concentration at the leading meniscus is thus not so pronounced.

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 Tube diameter $D = 0.02 \text{ m}$

• $(7.5 - 10) \cdot 10^{-2}$
 • $(2.5 - 5) \cdot 10^{-2}$
 • $(1.3 - 2.5) \cdot 10^{-2}$

The meniscus effect is due to two causes: the flow field and particle behaviour near the meniscus and the radial particle migration in the parallel flow at a distance from the meniscus.

In paper II the flow field near the leading meniscus is investigated. This flow, relative to the axis moving with the meniscus, is shown in fig 4 for tube Reynolds number 100 in a circular cylinder. It is seen that far behind the meniscus the flow is directed towards or away from the meniscus, depending on whether r/R is less or greater than $1/\sqrt{2}$. It was found by Karnis and Mason [4] that a spherical particle is affected by the walls such that it does not follow the streamlines. When the particle reverses its direction it returns closer to the axis than the corresponding streamline. This results in a net movement of particles towards the leading meniscus, thus causing the observed increase in particle concentration.

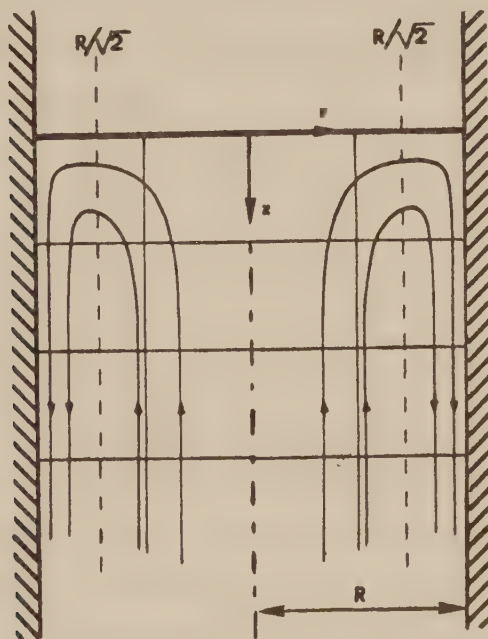


FIGURE 4. Streamline pattern near the leading meniscus in a circular cylinder for tube Reynolds number 100.

The radial particle migration in the parallel flow, which occurs at higher Reynolds numbers, causes the particle in the flow directed from the leading meniscus to move towards

the centre of the tube. In the centre of the tube the flow is directed towards the leading meniscus and a net increase in particle concentration in the region behind the meniscus occurs.

The study in paper II was undertaken in order to obtain information about the flow field behind the meniscus at Reynolds numbers apart from zero. Streamline patterns are obtained by numerically solving the equations of motion for rather high Reynolds numbers. Calculations are performed for both the axisymmetric (circular cylinder) and the two-dimensional case (two parallel plates).

The calculated streamline patterns at low Reynolds numbers are also compared with experimentally obtained patterns. Very small polystyrene particles were suspended in a fluid and their motion behind a meniscus was filmed. The agreement between calculated and experimentally obtained patterns is good. Discrepancies are attributed in part to the assumed boundary condition of a flat meniscus, which can hardly be achieved experimentally.

Papers III - IV

The major mechanism for the meniscus effect seems to be the radial particle migration, which occurs in flow with inertia effects. For zero Reynolds number (creeping motion or Stokes flow) the non-linear fluid-inertia term in the Navier-Stokes equation vanishes, and it can be proved that no radial movement of a neutrally bouyant particle can occur in Stokes flow. The theoretical interpretations of the radial particle migration are fairly few. This is due to the difficulty in solving the equations of motion when the non-linear inertia term is taken into account. Furthermore, theoretical investigations of particles suspended in flow confined between walls are restricted to very low Reynolds numbers. The studies reported in papers III - IV were carried out in order to examine the wall effects on a suspended particle when inertia effects are not negligible.

Two different geometries were studied:

1. The flow around a circular cylinder between two parallel walls
2. The flow around a sphere in a circular cylinder

The mathematical model is the same for both cases

$$\nabla p - \mu \nabla^2 \bar{\mathbf{v}} + \rho \bar{\mathbf{v}} \cdot \nabla \bar{\mathbf{v}} = 0$$

$$\nabla \cdot \bar{\mathbf{v}} = 0$$

with the boundary conditions

$$\bar{\mathbf{v}} = 0 \text{ on } S_p \text{ and } S_w$$

$$\bar{\mathbf{v}} = \bar{\mathbf{U}} \text{ at infinity}$$

S_p is the surface of the particle and $\bar{\mathbf{U}}$ is the Poiseuille velocity field.

The velocity field is expressed in terms of the streamfunction, ψ , and the vorticity, ω . Solutions are obtained for low and intermediate Reynolds numbers by numerically solving the resulting differential equations with the appropriate boundary conditions.

Streamlines, isovorticity lines and isobars have been calculated: see fig 5. The wall effects were examined by comparing the calculated drag force on the particle in proximity of walls with that for a particle in an unbounded flow. The agreement with analytically obtained results is very good at Reynolds number zero. Furthermore, the additional pressure drop and the additional wall friction force caused by the presence of the particle were calculated. The reliability of the data was checked by a momentum balance, which is very well satisfied for all Reynolds numbers treated.

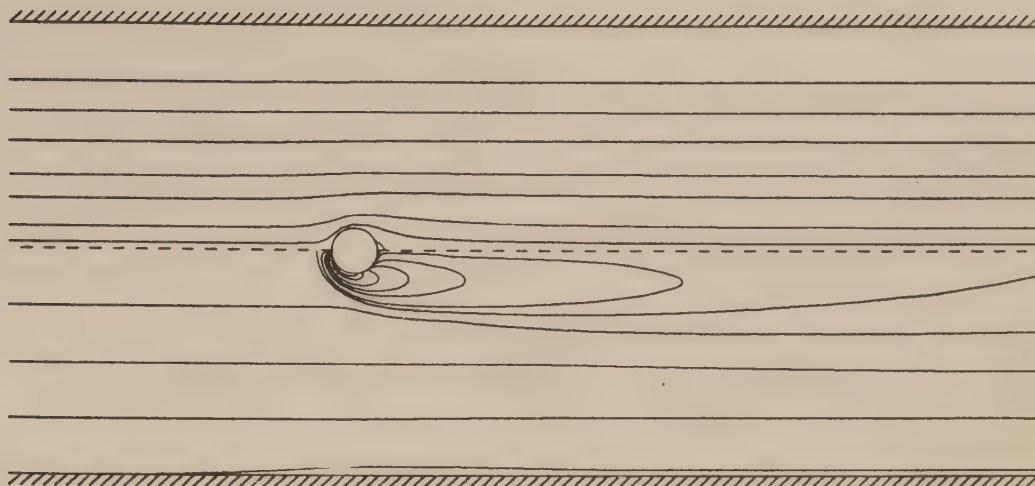


FIGURE 5. Streamlines (upper half) and isovorticity lines (lower half) for Poiseuille flow around a sphere in a circular cylinder for tube Reynolds number 150.

ACKOWLEGEMENT

I wish to express my gratitude to all friends and colleagues at the Department of Chemical Engineering for valuable assistance during the course of this work.

I also want to thank the staff at Lund University Computing Centre for helping me to interpret the most ambiguous error messages from the computer.

I am very indebted to tekn.lic. Dan Loyd for many stimulating discussions.

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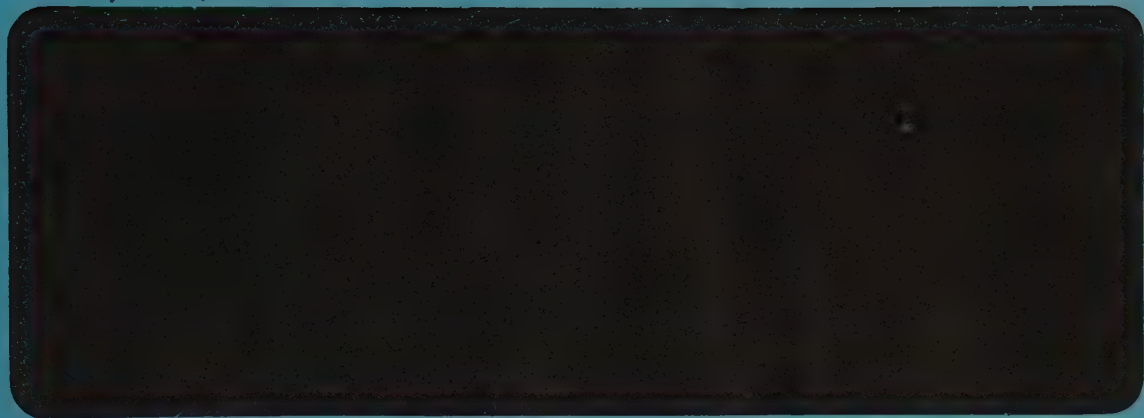
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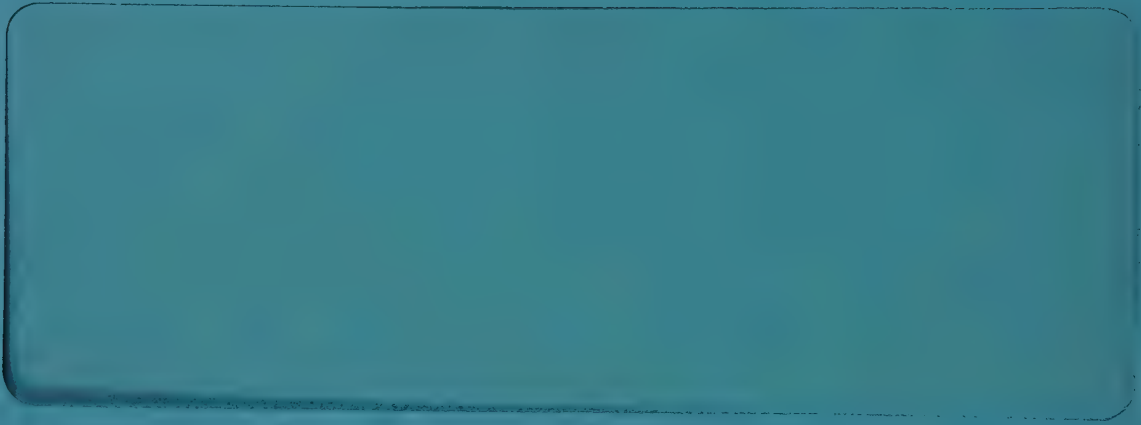
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Radial particle migration in plug flow — a method for solid-liquid separation and fractionation

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(First received 30 June 1969; in revised form 8 September 1969)

Abstract—A new method of separation for solid phase-liquid suspension is presented. The method is based on the radial particle migration, when the suspension is flowing in discrete plugs in tubes. The separation effect of various parameters is examined and it is found that the separation is favoured by increased velocity, increased ratio particle/tube diameter and reduced initial particle concentration. The process is also adaptable for fractionation due to the fact that the separation effect is dependent of the ratio particle/tube diameter.

INTRODUCTION

THE PURPOSE of this investigation is to demonstrate how the radial migration of particles and the meniscus effect can be exploited to separate and fractionate the particles in a suspension. If the suspension is allowed to flow through pipes in discrete plugs a reduction of concentration will be obtained at the trailing end of the plug and an increase at the leading end. By chopping-off the plug at a suitable point one obtains two flows: one with higher concentration than the original, one with lower.

Variables affecting separation are particle diameter, conduit diameter, plug velocity, plug length, travel length of the plug, viscosity of the liquid, density of particles and liquid, and concentration of particles.

LITERATURE

Segré and Silberberg[1, 2] observed that a spherical particle in a Poiseuille flow in pipes with low Reynolds number migrated radially across the flow path until reaching a state of equilibrium at 0.63 pipe radius from axis. This occurred regardless of whether the particle was introduced near the wall or the axis. The density of particle and liquid was identical. This radial migration has subsequently been found to be relatively independent of the Reynolds number and the number and the ratio of the sphere to

tube radius. On the other hand, the position of equilibrium seems to be dependent on the difference in density between the particle and the liquid.

The subject has been dealt with theoretically by Rubinow and Keller[3], Repetti and Leonard [4], Saffman[5] and, latest by Cox and Brenner [6]. An excellent summary of the research into the radial migration of particles in a Poiseuille flow, with a critical analysis of theories, has been prepared by Brenner[7].

The radial migration also explains the phenomenon that, if a particle suspension flows through a pipe from one vessel to another, the concentration of particles will be lower in the pipe than in the vessels. Seshadri and Suter[8] explain this by suggesting that radial migration drives the particles inwards so that they attain a higher mean velocity than the liquid. The higher mean velocity of the particles may be utilised by having the particles flow in discrete plugs, when the particles should migrate to the leading meniscus. That a concentration of particles will occur at the leading meniscus was first reported by Vejens [9], who found that the concentration of red corpuscles increased behind the leading meniscus when blood was drawn through capillary tubes. Forgacs *et al.*[10] carried out tests with a fibre suspension in a half-filled, vertically rotating glass tube. They found that the fibres collected

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at the leading meniscus. Karnis and Mason[11] have carried out thorough studies of flow conditions immediately behind the leading meniscus. They carefully studied the streamline pattern near the menisci with the aid of small (25μ) particles of aluminium.

The streamlines make a loop near the advancing liquid surface, which is shown in their investigation. This is also predicted by equations of Bhattacharji and Savic[12]. A streamline located at a distance β_1 ($\beta_1 < 1/\sqrt{2}$) from the axis will be at a distance β_2 after the loop ($\beta_2 > 1/\sqrt{2} > \beta_1$).

Karnis and Mason[11] also investigated the paths of spherical particles ($d/D = 0.003-0.213$) near the leading meniscus. They found that when a particle approaches the leading meniscus at a distance β_1 from the axis it does not follow the corresponding streamline. The particle is after the loop at a distance β_3 , where $\beta_1 < 1/\sqrt{2} < \beta_3 < \beta_2$. The difference $\Delta\beta = \beta_2 - \beta_3$ would therefore seem to imply that near the leading meniscus there is a net radial movement of liquid outwards. This results in a rearward movement of liquid with lower particle concentration than that arriving at the meniscus. The effect is thus a concentration of particles at the leading meniscus. Moreover it was found that small particles followed the flow lines more faithfully than large ones. Large particles should therefore be more likely to concentrate than small particles.

THEORY

For calculations and the presentation of results "separation diagrams", "concentration diagrams" and "z diagrams" are prepared.

Assume that a plug as per Fig. 1, with length L , contains the quantity of M particles. Introduce length coordinate l as per the figure. We now

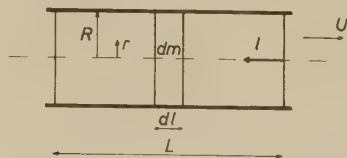


Fig. 1. Section through the flowing suspension plug.

define our positional coordinate or volume quotient x as per:

$$x = \frac{l}{L}. \quad (1)$$

If the quantity dm is located in a unit volume of the plug we define the mass quotient as per:

$$y = \frac{1}{M} \int_{l=0}^l dm. \quad (2)$$

The separation diagram is defined when $y = f(x)$ and indicates the part of the total quantity which is located ahead of a certain section of the plug.

If the original concentration is c_0 then the local concentration c in the plug will be:

$$c = c_0 \frac{dy}{dx} \quad (3)$$

since

$$c_0 = \frac{M}{A \cdot L}$$

and we obtain

$$c_0 \frac{dy}{dx} = \frac{M}{A \cdot L} \cdot \frac{dm/M}{dl/L} = \frac{dm}{A \cdot dl} = c.$$

The concentration diagram may be of either of two types: showing local concentration c as a function of x , or relative concentration c/c_0 as a function of x .

If we assume a mixture of several sizes of particles, the relative frequency for particle size $1 \leq i \leq k$ is defined:

$$z_i = \frac{dy_i}{dy}. \quad (4)$$

z_i is plotted against x and is thus the relative frequency of particle i within a given part of the plug. The z diagram is used for calculations in connection with fractionating and gives a good idea of the success of fractionating.

EXPERIMENTAL EQUIPMENT

The investigation was carried out with an apparatus, Fig. 2, which divides a particle sus-

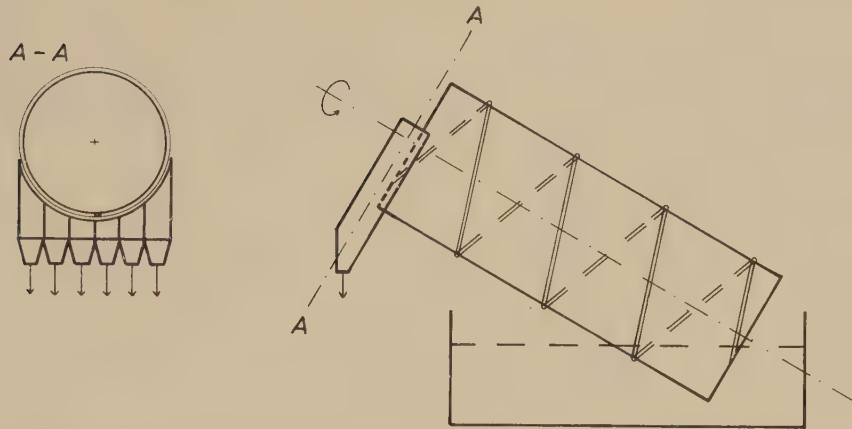


Fig. 2. The apparatus used for the experiments. It consists of a tube wound in spiral round a rotation drum. When the drum is rotating the tube collects a plug of the suspension, transports it to the vessels where the plug can be analysed.

pension into discrete plugs. The apparatus comprises a tube wound in a spiral round an inclined, rotating drum. The lower end of the drum is partly submerged in a particle suspension and when the drum rotates a plug will therefore be formed on each revolution. The plug flows to the upper end of the drum, where it is collected in a vessel divided into several compartments, and is analysed. Entrance effects are not yet examined and the original concentration, c_0 , is measured as the average concentration in the plug.

The tests were carried out with polystyrene spheres ($\rho_p = 1050 \text{ kg/m}^3$) in a mixture of water and glycerine with a density adjusted to $\rho_v = 1050 \text{ kg/m}^3$. ($\nu = 1.67 \cdot 10^{-6} \text{ m}^2/\text{sec}$.)

Test parameters

$d \cdot 10^3 \text{ m}$	$D \cdot 10^3 \text{ m}$	$u \text{ m/sec}$	$c_0 \text{ g/l}$	$L \text{ m}$	$S \text{ m}$
1.5–2.0	10		10		
0.75–1.0	20	0.2–0.6	50	0.35	7.20
0.25–0.50	30		100		

Mass concentration corresponding to volume concentration 0.01, 0.05 and 0.10, respectively.

The tube was wound $4\frac{1}{2}$ times round the drum, which had a diameter of 0.5 m. The suspension was kept well mixed by a suitable located vane on the drum. At the same time, the suspension was circulated by a pump which could also move

suspension from the outlet back to the lower vessel. At steady state, each compartment of the receiving vessel was analysed individually to determine the volume and the quantity of particles. Altogether some 300 tests were carried out, with good reproducibility. It is not possible to report all tests here, but merely a few chosen at random.

DISCUSSION OF TEST RESULTS

(a) Relationship between separation and velocity

Figure 3 shows the particle concentration in the plug as a function of x , with plug velocity as a parameter and other conditions constant.

At low velocities the separation is relatively poor, but quickly increases in the initial stages of velocity increase. In Fig. 4 the separation $y(x)$ is plotted as a function of Reynolds number given constant d/D . Olgård[13] carried out similar tests with fibres and got at small Reynolds numbers a very good separation, which then decreased as the Reynolds numbers increased.

Figure 5 shows the separation curves for various velocities. The diagonal in Fig. 5 indicates that no separation occurs in the plug, while the separation curves show that for a given $x = x_1$ a y_1 greater than x_1 was obtained, which indicates that we obtained an increase in concentration at the leading end of the plug.

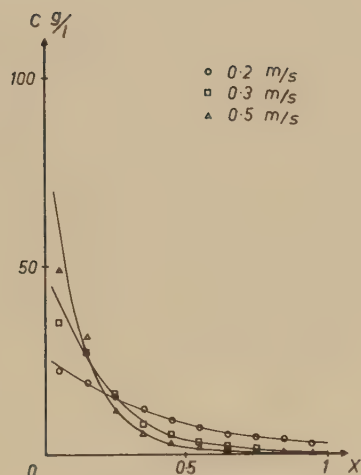


Fig. 3. Particle concentration in a streaming liquid plug as a function of the distance from the leading meniscus/the plug length ($x=l/L$) with the velocity as parameter. $c_0 = 10$ g/l. $d = (0.75-1.0) \cdot 10^{-3}$ m. $D = 2.0 \cdot 10^{-2}$ m.

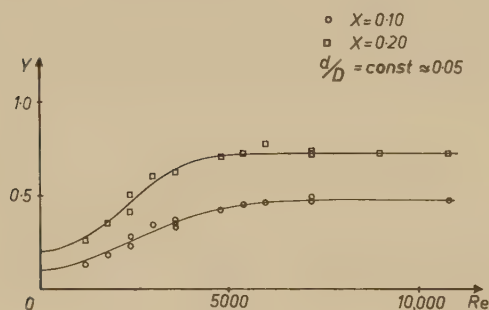


Fig. 4. The separation measured as y -value for $x = 0.10$ resp. 0.20 e.g. the part of the total quantity of particles, which is located ahead of a section 0.1 resp. $0.2 \times$ the plug length from the leading meniscus, as a function of the Reynold number. The ration particle/tube diameter (d/D) = 0.05 , $c_0 = 10$ g/l.

(b) Relationship between separation and particle diameter

Separation is greatly dependent on particle diameter, as seen from Fig. 6. Karnis and Mason found in their tests that the small particles followed the flow paths at the meniscus more faithfully than the large particles, and therefore were less inclined to concentrate. It has also been

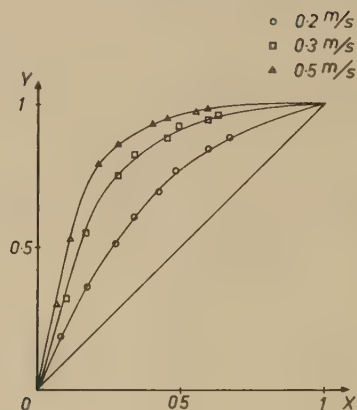


Fig. 5. Separation diagram showing the part of the total quantity of particles, which is located ahead of a certain section of the plug, with the velocity as a parameter. $c_0 = 10$ g/l, $d = (0.75-1.0) \cdot 10^{-3}$ m, $D = 2.0 \cdot 10^{-2}$ m.

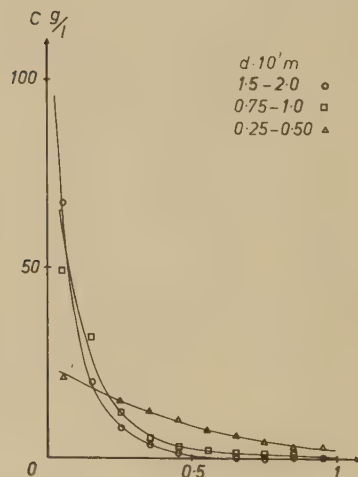


Fig. 6. The particle concentration in the streaming plug with the particle diameter as a parameter. $c_0 = 10$ g/l, $u = 0.5$ m/sec, $D = 2.0 \cdot 10^{-2}$ m.

found that the radial migration rate of the particles increases with increased ratio sphere/cylindrical radius. The position of equilibrium, on the other hand, is unaffected by particle size. Olgård's tests[13] with fibres also showed that separation is better for large fibres than for small. We shall show later on how this can be exploited in fractionating.

(c) Relationship between separation and tube diameter

Figure 7 shows that the tube diameter has a significant effect on separation. A large d/D ratio is required for good separation.

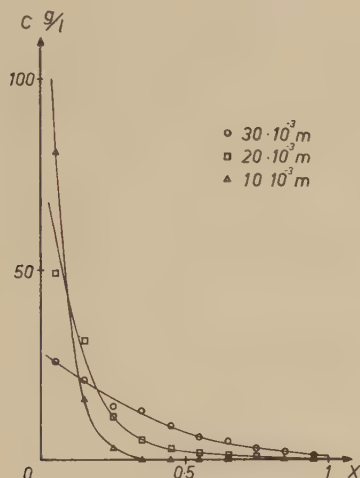


Fig. 7. The particle concentration in the streaming plug with the tube diameter as a parameter. $c_0 = 10$ g/l, $u = 0.5$ m/sec, $d = (0.75-1.0) \cdot 10^{-3}$ m.

(d) Relationship between separation and concentration

The highest concentration obtained in the leading part of the plug is about 500 g/l, corresponding to a volumetric concentration of 0.5, regardless of particle size. Since the maximum concentration is not exceeded, a particle plug of constant concentration is formed at the leading end. Therefore, the higher the initial concentration the lower will be the increase in concentration, but there will be a long zone with constant concentration. See Fig. 8. When high concentrations of particles occur in the leading end of the plug the effective viscosity increases, resulting in a rearward displacement of the "effective" meniscus when the viscosity is sufficiently high.

FRACTIONATION

The relationship between separation and particle size opens a possibility of fractionating. Tests indicate that this can be done successfully.

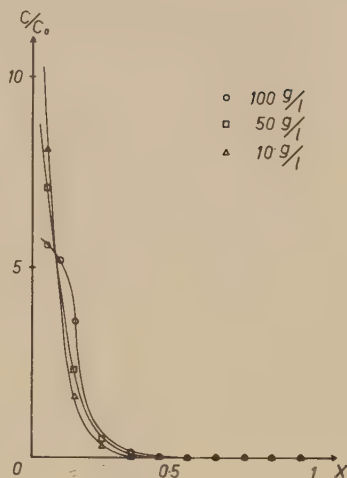


Fig. 8. The relative concentration (the local concentration/the initial concentration) in a streaming plug with the initial concentration as a parameter. $u = 0.5$ m/sec, $D = 2.0 \cdot 10^{-2}$ m/sec, $d = (0.75-1.0) \cdot 10^{-3}$ m/sec.

For low particle concentrations in which all particles are free to move in the suspension it is possible to calculate, from each particle size's separation curve, the total separation curve for a suspension comprising different sized particles. For a given particle size, Eq. (2) applies:

$$y_i = \frac{1}{M_i} \int_{l=0}^l dm_i$$

whereby:

$$y = \frac{1}{M} \sum_{i=1}^k \int_{l=0}^l dm_i \quad (5)$$

where

$$M = \sum_{i=1}^k M_i. \quad (6)$$

In one test, three sizes of particle were employed—see Table 1. The table shows that the y values for each particle size obtained during the fractionation trial agree very closely with the y values for the separation of the particles individually. The calculated total curve agrees closely with the experimental curve. Figure 9 comprises the experimental and calculated z curves.

Table 1. Comparison of calculated and measured y - and z -values: $D = 2 \cdot 10^{-2}$ m; $u = 0.3$ m/sec; $c_{01} = c_{02} = c_{03} = 50/3$ g/l; $d_1 = (1.5-2.0) \cdot 10^{-3}$ m; $d_2 = (0.75-1.0) \cdot 10^{-3}$ m; $d_3 = (0.25-0.50) \cdot 10^{-3}$ m

x	d_1		d_2		d_3		Calc.		Exp.	Calc.			Exp.		
	y_1	y_{1exp}	y_2	y_{2exp}	y_3	y_{3exp}	y	y		z_1	z_2	z_3	z_1	z_2	z_3
0.1	0.58	0.62	0.30	0.29	0.13	0.12	0.34	0.34		1.72	0.89	0.39	1.83	0.85	0.35
0.2	0.84	0.94	0.54	0.52	0.26	0.24	0.55	0.56		1.24	1.14	0.62	1.45	1.05	0.54
0.3	0.94	0.98	0.70	0.72	0.38	0.38	0.67	0.69		0.79	1.26	0.95	0.31	1.54	1.04
0.4	0.98	1.0	0.81	0.82	0.50	0.50	0.76	0.77		0.45	1.22	1.33	0.25	1.25	1.50
0.5	0.99	1.0	0.88	0.89	0.61	0.62	0.83	0.84		0.16	1.11	1.73	0	1.00	1.83
0.6	1.0	1.0	0.93	0.93	0.71	0.71	0.88	0.88		0.19	0.94	1.87	0	1.00	2.25
0.7	1.0	1.0	0.96	0.96	0.79	0.80	0.92	0.92		0	0.82	2.18	0	0.75	2.25
0.8	1.0	1.0	0.98	0.98	0.87	0.87	0.95	0.95		0	0.60	2.40	0	0.67	2.33
0.9	1.0	1.0	0.99	0.99	0.94	0.95	0.98	0.97		0	0.38	2.62	0	0.33	2.67
1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0		0	0.43	2.57	0	0.50	2.50

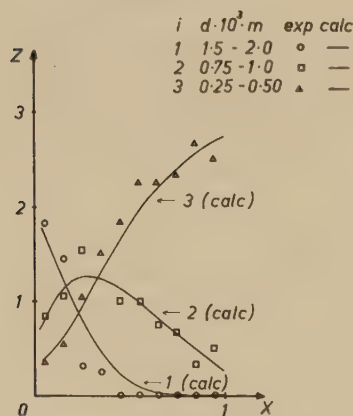


Fig. 9. z diagram showing the relative frequency of three particle sizes in the streaming plug. See Table 1.

By means of the diagram, and with knowledge of the composition of the suspension (through fractionation) it is possible to calculate the local composition of each part of the plug. The percentage portion of each component in the suspension can be expressed:

$$P_i = \frac{M_i}{M} \cdot 100. \quad (7)$$

In an arbitrary part of the plug, therefore, the percentage portion of the component i will be (by definition):

$$P_{ii} = \frac{dm_i}{dm} \cdot 100. \quad (8)$$

Since $z_i = dy_i/dy$ we obtain

$$P_{ii} = z_i \cdot P_i. \quad (9)$$

As an example, the composition of the plug between e.g. $x = 0.2-0.3$ is estimated.

Original composition:

$$P_1 = P_2 = P_3 = 100/3 \text{ per cent.}$$

From Fig. 7 we obtain (experimental value):

$$z_1 = 0.51; z_2 = 1.52; z_3 = 0.96.$$

The composition will thus be:

$$P_{11} = 17 \text{ per cent; } P_{12} = 51 \text{ per cent; } P_{13} = 32 \text{ per cent.}$$

Using the z diagram, we can thus calculate the particle composition at each part of the plug. By fractionation in an additional stage it is possible to increase the percentage of the required fraction yet further. If it occurs that the z curves intersect at $z = 1$ the best thing is to recirculate that portion of the plug since it then must have the same composition as the original suspension.

DISCUSSION

In the introduction a note was made of the variables affecting separation. In this investigation we have chosen to study the effect of the velocity of the plug, particle diameter, tube diameter and particle concentration. Further investigations are in hand, where the specific significance of plug length, travel length of the

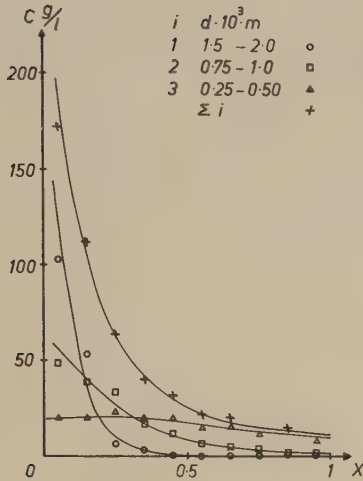


Fig. 10. The particle concentration in the streaming plug from a fractionation experiment. $D = 2.0 \cdot 10^{-2}$ m, $u = 0.3$ m/sec, $c_{01} = c_{02} = c_{03} = 50/3$ g/l, $c_{tot} = 50$ g/l.

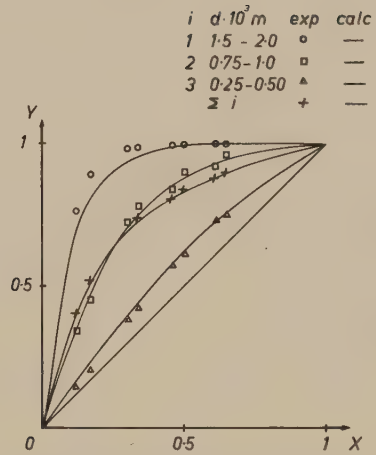


Fig. 11. Separation diagram showing both measured and calculated values from Table 1.

plug, and differences in density between particles and liquid are being studied. Briefly, one may describe the effect of the four variables hitherto examined as follows:

Separation is favoured by:

- increased velocity (applies at low velocities)
- increased particle diameter
- reduced tube diameter
- reduced concentration.

Figure 4 is particularly interesting. This shows the degree of separation as a function of Re at constant d/D .

For fibres, Olgård[13] obtains extremely good separation at very low Re values, though a deterioration was noted when Re increased. For spherical particles, however, separation is poor when Re is low, but improves as Re increases. The difference in behaviour is due to the fact that, given low Re , a strong interaction occurs in fibre suspensions, which causes the fibres to floc. This effect dominates at low Re values and is of less importance when Re is higher. Spherical particles have far less mutual effect and the separation process is due exclusively to radial migration of particles. Further tests with Re above 10,000 will be carried out, in which con-

figuration other than spheres will be employed.

The mechanism of separation is still somewhat unclear, but it is perfectly plain that radial migration plays the major role. The flow conditions near the two menisci are being studied and a further report will be published on this.

NOTATION

- A cross sectional area of tube
- c particle concentration
- d particle diameter
- D tube diameter
- l distance from leading meniscus
- L length of plug
- M quantity of particles
- P percentage portion
- Re Reynolds number $= (u \cdot D)/\nu$
- r distance from tube axis
- R tube radius
- S travel length of plug
- u plug velocity
- x volumetric coefficient as per Eq. (1)
- y mass coefficient as per Eq. (2)
- z relative frequency of particle size as defined by Eq. (4)

Greek symbols

- β ratio r/R
- ρ density

ν viscosity k number of components

Subscripts

 l local value i component number 0 initial value

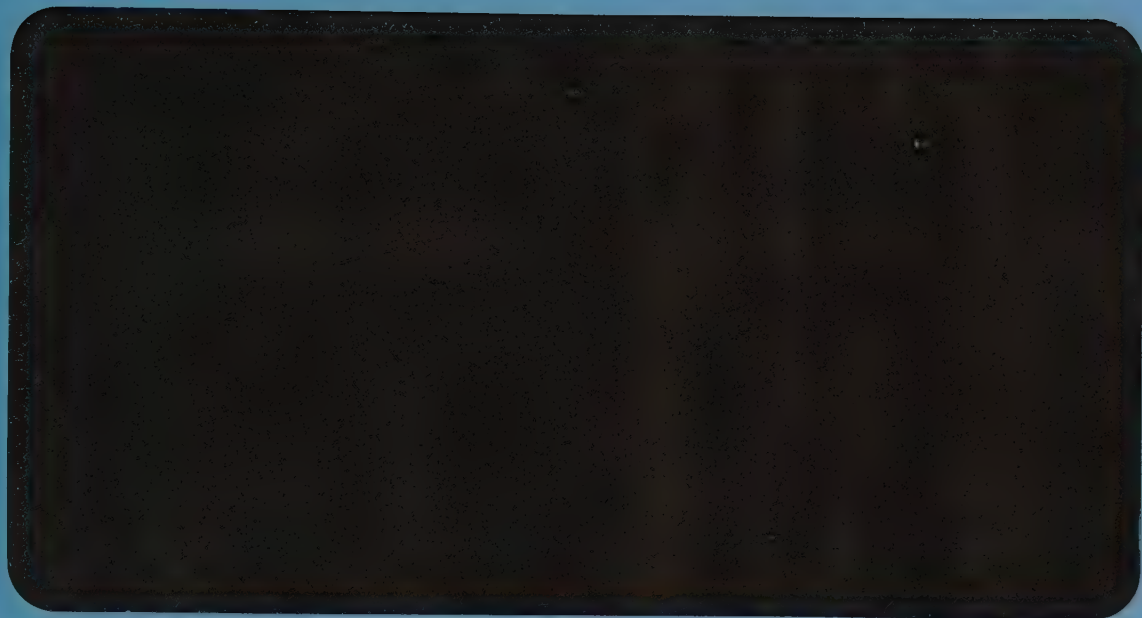
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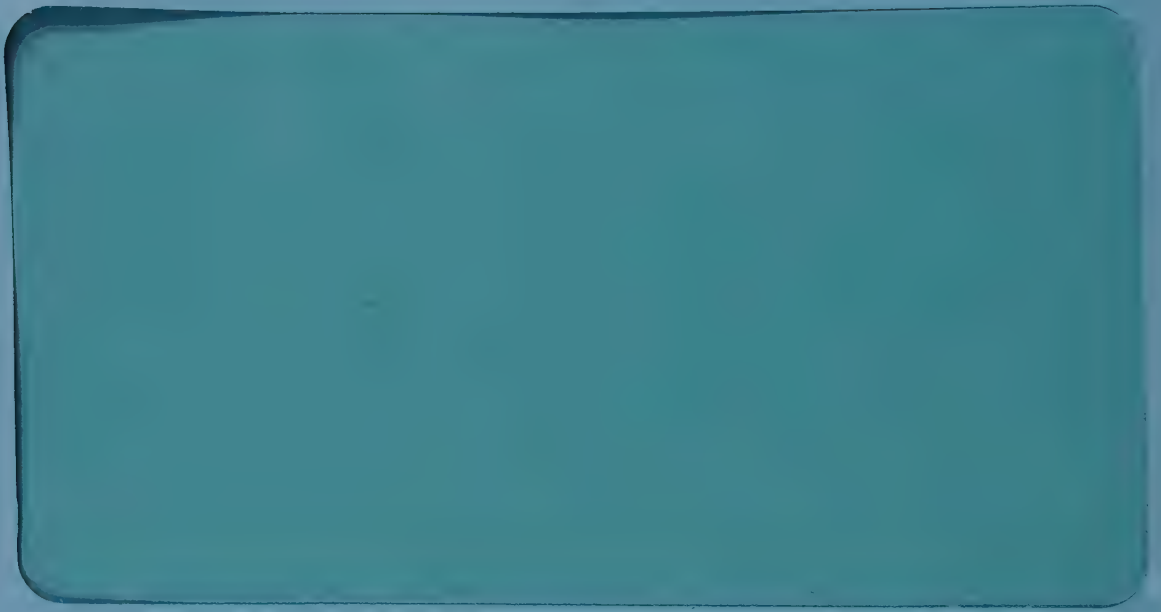
Résumé — Une nouvelle méthode de séparation pour une suspension liquide-phase solide est présentée. La méthode repose sur le principe de la migration radiale des particules quand la suspension s'écoule en tampons séparés à l'intérieur de tubes. On étudie l'effet de séparation de paramètres variés et on trouve que la séparation est favorisée par une vitesse accrue, un rapport croissant entre le diamètre du tube et les particules et une concentration initiale réduite des particules. Le procédé s'adapte aussi au fractionnement, parce que l'effet de séparation dépend du rapport diamètre du tube/particule.

Zusammenfassung — Es wird über ein neues Verfahren zur Trennung fest-flüssiger Suspensionen berichtet. Das Verfahren beruht auf der radialen Teilchenwanderung, wenn sich die Suspension in getrennter Pfropfströmung in Röhren befindet. Die Trennwirkung der verschiedenen Parameter wird untersucht und es wird festgestellt, dass die Trennung durch erhöhte Geschwindigkeit, erhöhtes Teilchen-Rohrdurchmesser Verhältnis und verringerte Anfangskonzentration der Teilchen begünstigt wird. Das Verfahren ist auch auf die Fraktionierung anwendbar infolge der Tatsache, dass die Trennwirkung vom Teilchen-Rohrdurchmesserverhältnis abhängig ist.

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Shorter Communications

On the flow near a liquid surface in a vertical channel with moving walls

(Received 1 March 1971)

WHEN PARTICLE suspensions are located in certain shear-fields, the particles migrate across the streamlines[1, 2]. Moreover, particles accumulate behind an advancing air-liquid meniscus in a channel, (see, for instance, [3-5]). These phenomena can be exploited for separation and fractionation processes on an industrial scale. Results of earlier laboratory experiments have been reported by, e.g. Johansson, Olgård and Jernqvist[6].

The purpose of this paper is to report some calculations on the streamline pattern near the free liquid surface, the meniscus, in vertical channels with moving walls according to Figs. 1 and 2. Some comparisons have also been made between experimental and theoretical investigations. The flow, which has been studied for two-dimensional channels and cylindrical tubes, is assumed to be incompressible and steady. For those Reynolds' numbers, which have so far been studied, the flow is supposed laminar and symmetric about the centre of the channel.

For the two-dimensional channel the Navier-Stokes' equation in dimensionless quantities reads

$$\bar{\nabla}^2 \bar{\omega} + Re_L \left(\frac{\partial \bar{\psi}}{\partial \bar{x}} \cdot \frac{\partial \bar{\omega}}{\partial \bar{y}} - \frac{\partial \bar{\psi}}{\partial \bar{y}} \cdot \frac{\partial \bar{\omega}}{\partial \bar{x}} \right) = 0 \quad (1)$$

$$\bar{\nabla}^2 \bar{\psi} = -\bar{\omega} \quad (2)$$

where $\bar{\psi}$ is the streamfunction, $\bar{\omega}$ the vorticity and $\bar{\nabla}^2 = (\partial^2/\partial \bar{x}^2) + (\partial^2/\partial \bar{y}^2)$. The velocities are obtained by $\bar{u} = (\partial \bar{\psi}/\partial \bar{y})$ and $\bar{v} = -(\partial \bar{\psi}/\partial \bar{x})$ and thereby the equation of continuity is satisfied. The Reynolds' number, Re_L , is defined as $Re_L = U \cdot L/\nu$. According to Fig. 1 the boundary conditions for this case are:

$$\bar{x} = 0 \begin{cases} \bar{u} = 0 \\ \frac{\partial \bar{v}}{\partial \bar{x}} = 0 \end{cases} \quad \bar{x} \rightarrow \infty \begin{cases} \bar{u} = \frac{3}{2} \bar{y}^2 - \frac{1}{2} \\ \bar{v} = 0 \end{cases}$$

$$\bar{y} = 0 \begin{cases} \bar{v} = 0 \\ \frac{\partial \bar{u}}{\partial \bar{y}} = 0 \end{cases} \quad \bar{y} = 1 \begin{cases} \bar{u} = 1 \\ \bar{v} = 0. \end{cases}$$

For the cylindrical tube the corresponding equations are:

$$\bar{E}^2(\bar{r}\bar{\omega}) + \frac{1}{2} Re_D \left(\frac{\partial \bar{\psi}}{\partial \bar{r}} \cdot \frac{\partial \bar{\omega}}{\partial \bar{z}} - \frac{\partial \bar{\psi}}{\partial \bar{z}} \cdot \frac{\partial \bar{\omega}}{\partial \bar{r}} + \frac{\bar{\omega}}{\bar{r}} \cdot \frac{\partial \bar{\psi}}{\partial \bar{z}} \right) = 0 \quad (3)$$

$$\frac{1}{\bar{r}} \bar{E}^2(\bar{\psi}) = \bar{\omega} \quad (4)$$

where the operator $\bar{E}^2 = (\partial^2/\partial \bar{r}^2) - (1/\bar{r})(\partial/\partial \bar{r}) + (\partial^2/\partial \bar{z}^2)$. The

velocities are defined according to $\bar{v}_r = (1/\bar{r})(\partial \bar{\psi}/\partial \bar{z})$ and $\bar{v}_z = -(1/\bar{r})(\partial \bar{\psi}/\partial \bar{r})$. The Reynolds' number, Re_D , is defined $Re_D = U \cdot D/\nu$ (see Fig. 2). The boundary conditions in this case are:

$$\bar{z} = 0 \begin{cases} \bar{v}_z = 0 \\ \frac{\partial \bar{v}_r}{\partial \bar{z}} = 0 \end{cases} \quad \bar{z} \rightarrow \infty \begin{cases} \bar{v}_z = 2\bar{r}^2 - 1 \\ \bar{v}_r = 0 \end{cases}$$

$$\bar{r} = 0 \begin{cases} \bar{v}_r = 0 \\ \frac{\partial \bar{v}_z}{\partial \bar{r}} = 0 \end{cases} \quad \bar{r} = 1 \begin{cases} \bar{v}_z = 1 \\ \bar{v}_r = 0. \end{cases}$$

A finite-difference method is used for the calculations. The number of netpoints is maximally 4221 and the mesh size is constant throughout the field. The differential equations for $\bar{\psi}$ and $\bar{\omega}$ are transformed to a system of difference equations. The main diagonals in the matrices, which arise in the calculations, are dominant. The method of solution is iterative and a method with overrelaxation is used[7, 8]. The computations for the two-dimensional case[9] were carried out on the computer IBM 360/65 at Chalmers University of Technology, Göteborg, and for the cylindrical case the computer UNIVAC 1108 at Lund University Computing Centre, Lund was used. The computer time for one Reynolds' number is 2-3 min in the two-dimensional case and somewhat longer in the axisymmetrical.

The streamline pattern for the axisymmetrical case has been studied experimentally. The methods and equipment were similar to that used by Karnis and Mason[5]. The experiments were carried out in a vertical glass tube with inner diameter 7.35×10^{-3} m. The streamline pattern behind the meniscus was obtained by filming small polystyrene particles ($\rho_p = 1050$ kg/m³) in the liquid. In order to reduce optical distortion the glass tube was surrounded by a square "Perspex" box containing the same liquid as inside the tube. Furthermore the liquid and the glass had the same refractive index. The solution consisted of a mixture of polypropylene-glycol (Berol FTM-74) and dibutylphthalate. The kinematic viscosity of the mixture, ν , was 76.3×10^{-6} m²/sec and the density, ρ_l , 1027 kg/m³ at 20°C.

The experimentally determined streamline patterns for the flow at low Reynolds' numbers were essentially independent of the Reynolds' number. The calculations for the pure liquid show the same result and the correlation between calculations and experiments is good. In Fig. 3 some of the calculated streamlines for the two-dimensional channel at Reynolds' numbers 0, 1 and 10 are shown. The computed results give nearly identical streamline patterns. This behaviour applies also for the axisymmetrical case.

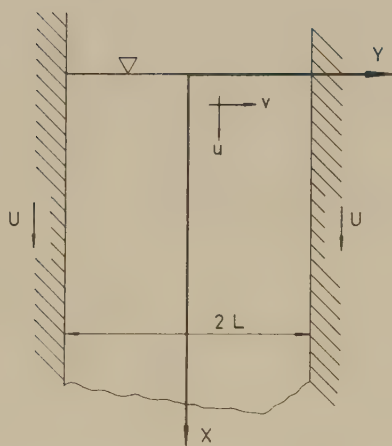


Fig. 1. Geometry and coordinate system for the two-dimensional channel.

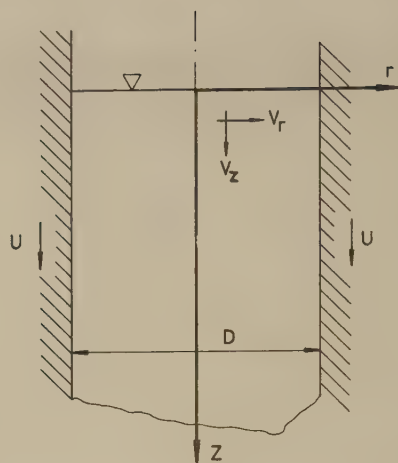


Fig. 2. Geometry and coordinate system for the axisymmetric pipe.

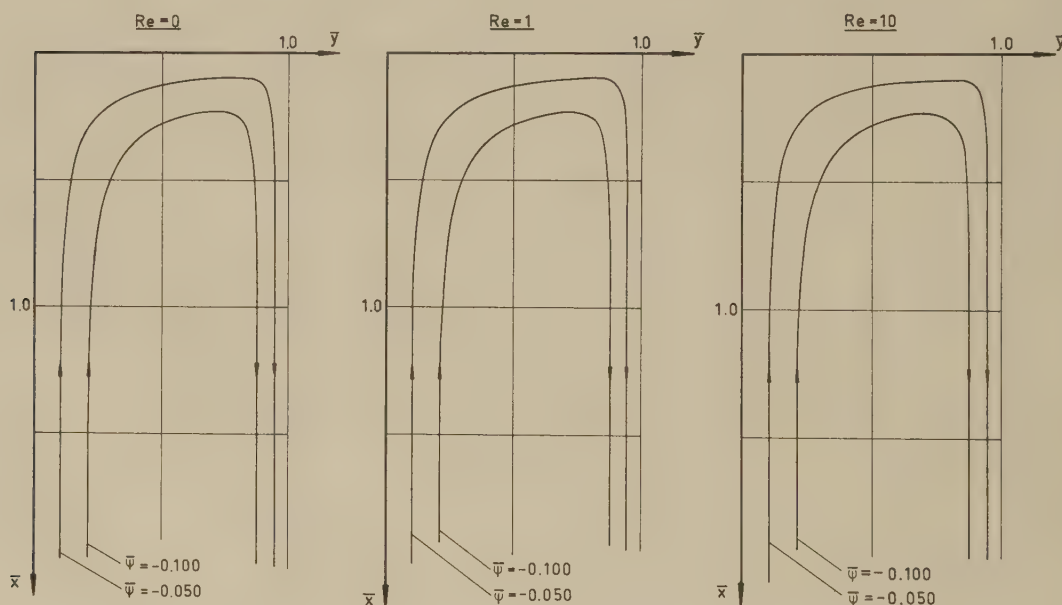


Fig. 3. Schema of the flow pattern near the meniscus in the channel for $Re_L = 0, 1$ and 10 .

At higher Reynolds' numbers, however, there is a difference between the calculated streamline patterns. This result is also obtained in a comparison between the experimentally determined streamline patterns. In Figs. 4 and 5 some computed streamlines are reproduced for the two-dimensional and axisymmetric case respectively with Reynolds' numbers 100 and 1000, respectively 100 and 500 and as a comparison the streamline pattern for $Re = 1$.

High Reynolds' numbers also increase the difference between the particle paths in the suspension and the stream-

lines in the pure fluid and this holds especially for the area close to the meniscus.

In Fig. 6 computed and filmed streamlines for the axisymmetric case with $Re_D = 1$ are shown. The mesh used is 21×201 . Reasons for the difference between the calculated and experimental values are i.a. influence of mesh size, particle size and that the calculations are based on the boundary condition for a flat liquid surface while the experiments give a curved surface near the wall because of the surface tension. Calculations have earlier been carried out

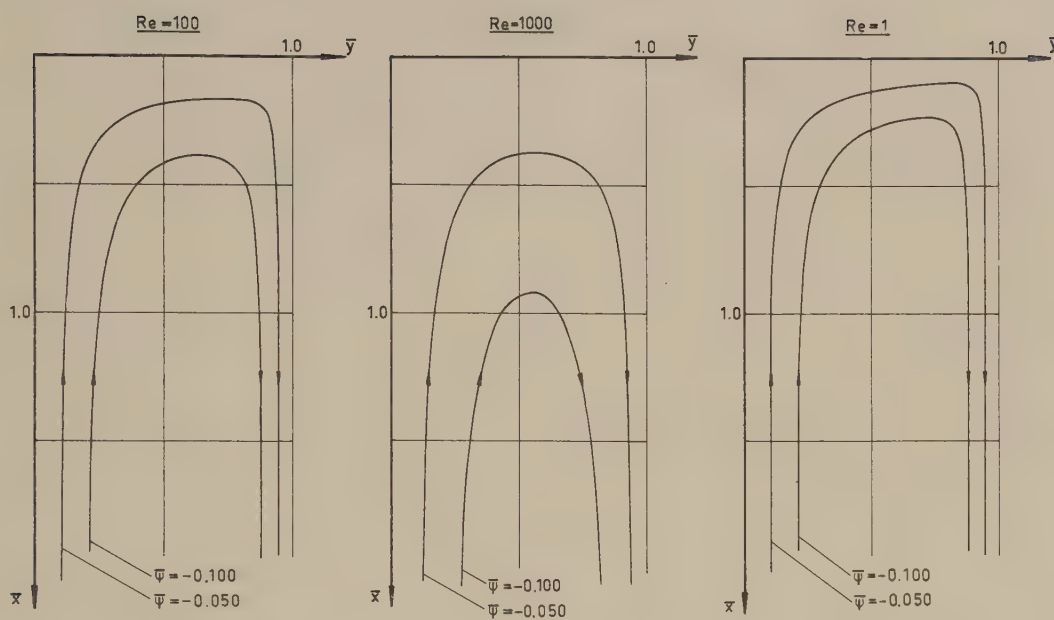


Fig. 4. Schema of the flow pattern near the meniscus in the channel for $Re_L = 100, 1000$ and 1 .

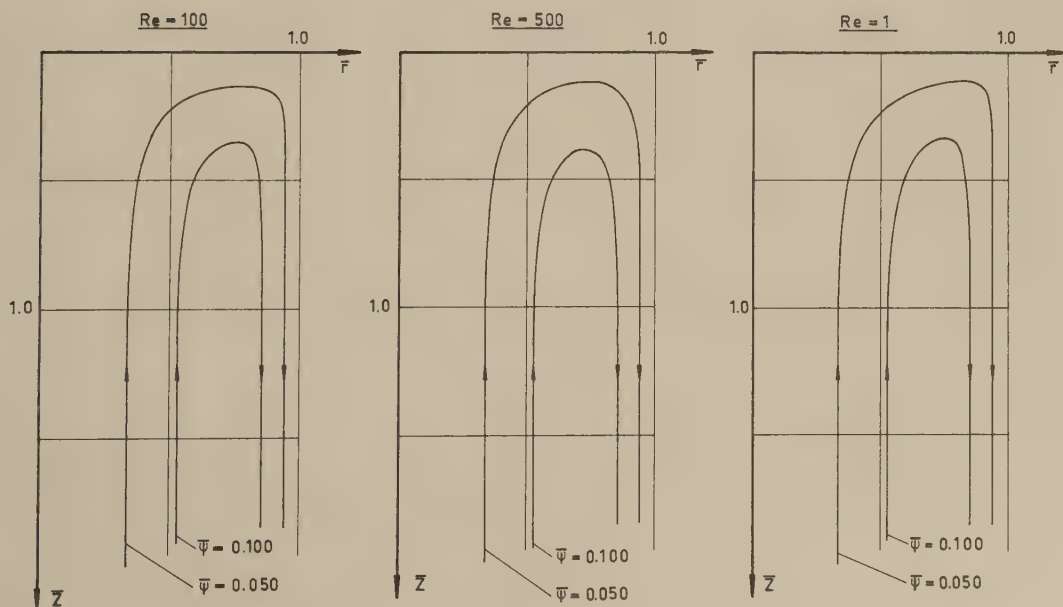


Fig. 5. Schema of the flow pattern near the meniscus in the pipe for $Re_D = 100, 500$ and 1 .

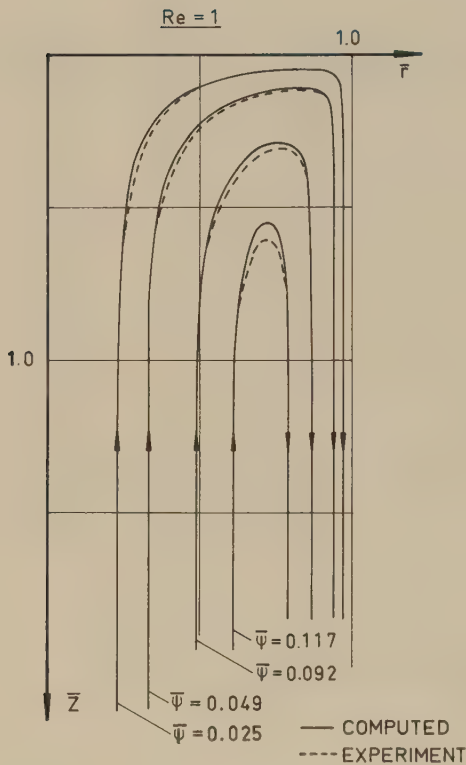


Fig. 6. Schema of the flow pattern near the meniscus in the pipe for $Re_D = 1$ showing both computed and experimental values.

by Bhattacharji and Savic[10], who solved the case for $Re = 0$ analytically, but computed the streamlines with a truncated series.

In the continued investigations the migration of a single particle in shearfields will be studied and from these results, together with those presented here, it should be possible to draw certain conclusions concerning the mechanism of the separation and fractionation processes mentioned.

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NOTATION

D	tube diameter
$\bar{E}^2$	operator, $\bar{E}^2 = \frac{\partial^2}{\partial \bar{r}^2} - \frac{1}{\bar{r}} \cdot \frac{\partial}{\partial \bar{r}} + \frac{\partial^2}{\partial \bar{z}^2}$
L	half channelwidth
$\bar{r}$	dimensionless radius, $\bar{r} = \frac{2 \cdot r}{D}$
Re	Reynolds' number
U	wall velocity
$\bar{u}$	dimensionless velocity in $\bar{x}$ -direction, $\bar{u} = \frac{u}{U}$
$\bar{v}$	dimensionless velocity in $\bar{y}$ -direction, $\bar{v} = \frac{v}{U}$
$\bar{v}_{\bar{r}}$	dimensionless velocity in $\bar{r}$ -direction, $\bar{v}_{\bar{r}} = \frac{v_r}{U}$
$\bar{v}_{\bar{z}}$	dimensionless velocity in $\bar{z}$ -direction, $\bar{v}_{\bar{z}} = \frac{v_z}{U}$
$\bar{x}$	dimensionless x -coordinate, $\bar{x} = \frac{x}{L}$
$\bar{y}$	dimensionless y -coordinate, $\bar{y} = \frac{y}{L}$
$\bar{z}$	dimensionless z -coordinate, $\bar{z} = \frac{2 \cdot z}{D}$
$\bar{\nabla}^2$	operator, $\bar{\nabla}^2 = \frac{\partial^2}{\partial \bar{x}^2} + \frac{\partial^2}{\partial \bar{y}^2}$

Greek symbols

$\bar{\psi}$	dimensionless streamfunction, $\bar{\psi} = \frac{\psi}{U \cdot L}$ resp. $\bar{\psi} = \frac{4 \cdot \psi}{U \cdot D^2}$
$\bar{\omega}$	dimensionless vorticityfunction, $\bar{\omega} = \frac{L}{U} \cdot \omega$ resp. $\bar{\omega} = \frac{D}{2 \cdot U} \cdot \omega$
ν	kinematic viscosity
ρ_l	density (liquid)
ρ_p	density (particle)

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A NUMERICAL SOLUTION OF THE FLOW
AROUND A CIRCULAR CYLINDER BETWEEN
TWO PARALLEL PLATES

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REPORT NO 74-F-2

The Poiseuille flow around a circular cylinder situated halfway between two parallel plates is theoretically investigated. The governing equations of motion are solved numerically for Reynolds numbers (based on the mean velocity in the channel and the channel width) up to 100.

The method of solution consists of obtaining two solutions for the flow field; one for an "inner" region close to the cylinder and one for an "outer" region far from the cylinder and close to the channel walls. The two solutions are computed with different coordinate systems in order to represent the boundary conditions properly; circular cylindrical coordinates in the "inner" field and rectangular coordinates in the "outer" field. In an iteration procedure these two solutions are matched and a solution for the whole flow field is achieved.

Streamlines, isovorticity lines and isobars are calculated and presented in figures. Drag coefficients, pressure drops, pressure distributions, forces on the channel walls etc. are computed. The agreement with analytically obtained results for Reynolds number zero is very good. The wall effects are examined by comparing the calculated drag force on the cylinder in proximity of walls to that for an unbounded fluid. It is found that the wall effects become negligible for Reynolds numbers beyond 100. The critical Reynolds number of separation is estimated to 45.5.

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APPENDIX 2: FLOW SHEET FOR THE VORTICITY AND
THE STREAMFUNCTION CALCULATIONS

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CULATIONS

1 INTRODUCTION

Among the problems that interested the earliest hydrodynamicists was the flow around immersed bodies. However the analytical solutions of the equations of motion met with great difficulties because of the non-linearity of the governing equations.

For low Reynolds numbers, when inertia forces are small compared with viscous forces, the non-linear terms in the governing equations can be omitted or modified. Complete omission of the inertia terms results in the equation of creeping motion or Stokes' equation (Reynolds number is zero). This equation was used by Stokes (1851) [1] to obtain a solution of the flow field due to a rigid sphere. For higher but still very small Reynolds numbers Stokes' solution leads to an inconsistency. The inertia forces are then of the same magnitude as the viscous forces at a distance from the sphere.

In order to improve Stokes' solution Oseen (1910) [2] took into account also the inertia terms, which he modified to still get a linear differential equation. Oseen's work became a base for many other investigators and Proudman and Pearson (1957) [3] obtained a solution for the whole flow field for Reynolds numbers apart from zero but still very small.

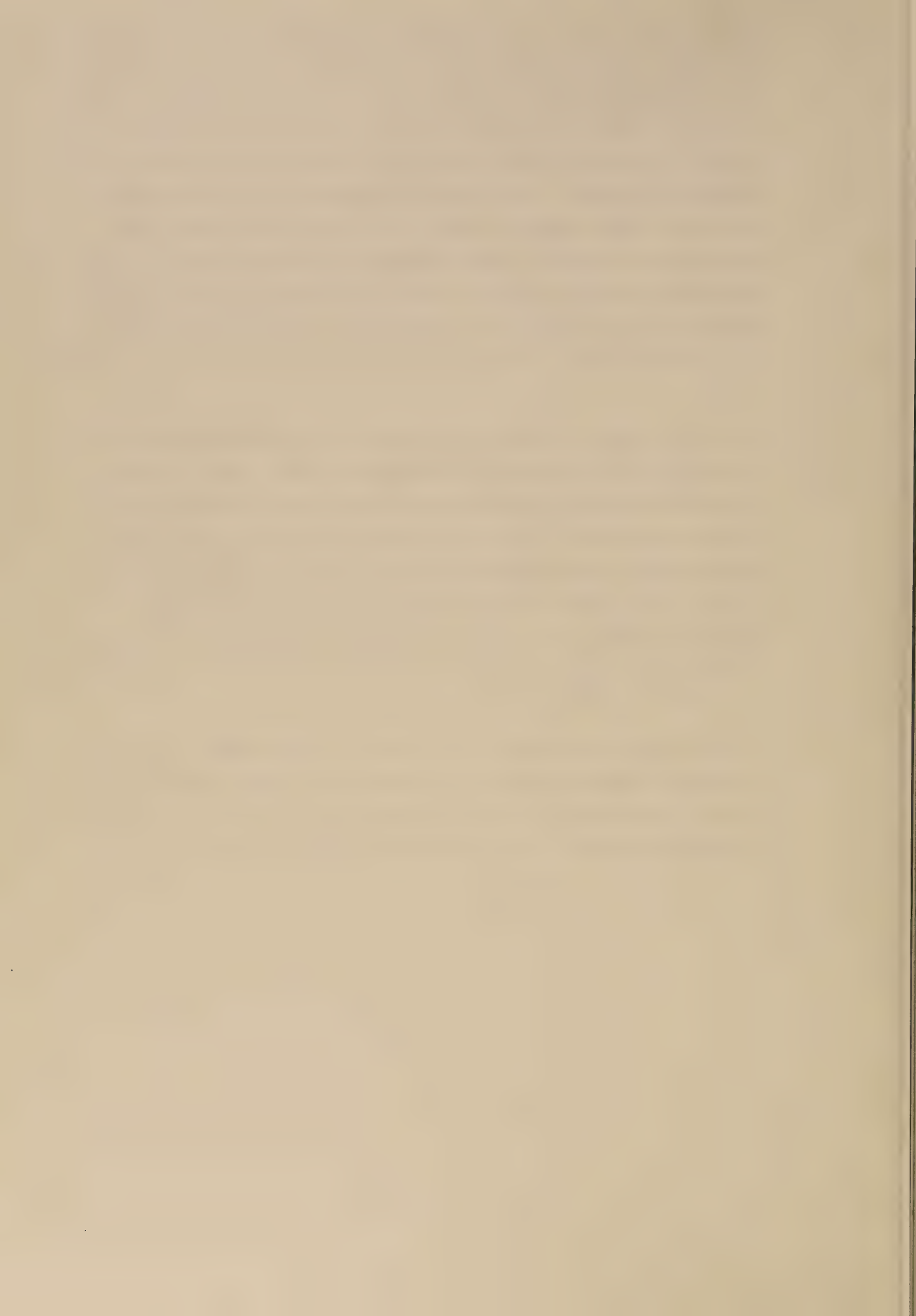
All body shapes are not tractable. For two-dimensional flow perpendicular to the axis of a circular cylinder there exists no solution of the equation of creeping motion (Stokes' paradox). However Lamb (1911) [4] has employed Oseen's method of solution to obtain an approximation for the drag per unit length of a cylinder.

For very high Reynolds numbers a common way of simplifying the equations of motion is complete omission of the viscous forces and assumption of irrotational motion. The equations of potential flow are then obtained. These equations unfortunately give no information on the drag experienced by the immersed bodies (d'Alembert's paradox), because the solutions can not be made to satisfy the no slip boundary condition.

In the case of large Reynolds numbers it is possible to divide the flow field into an external field, where the flow is irrotational, and an internal field near the body, where viscous effects are not negligible. The external field can be handled with the potential flow theory, whereas the inner field must be treated by some form of the equations of motion. The inner field consists of a thin boundary layer and a wake behind the body. The flow in the boundary layer can be calculated by a theory developed by Prandtl (1905) [5]. The boundary layer theory implies simplifications of the equations of motion, due to the fact that the boundary layer is thin. However, the boundary layer theory can not treat the flow in the wake behind a body.

A numerous amount of theoretical [6] - [37] (both analytical and numerical) and experimental investigations [38] - [51] of the unbounded flow around spheres and cylinders have been performed. The flow around bodies in proximity of walls has only to some extent been treated, and theoretically only for very low Reynolds numbers [52] - [66]. Among those who have dealt with the flow around a cylinder between two parallel plates are Faxen [62], [63] and Takaisi [64], [65]. A numerical solution is presented by Loyd [66] for Reynolds number zero.

This investigation deals with Poiseuille flow around a circular cylinder between two parallel plates for different Reynolds numbers. A numerical solution is presented and the complete equations of motion are used for an incompressible and viscous fluid.



2 DESCRIPTION OF THE PROBLEM

A circular cylinder is situated halfway between two parallel plates, with the cylinder immobile relative to the plates. The extension of the cylinder and of the plates is infinite in the z -direction, which leads to a two-dimensional problem. The origo of the coordinate system is placed at the centre of the cylinder, see fig 1. A flow with the velocity profile

$$v_x = \frac{3}{2} U \left(1 - \left(\frac{y}{L/2}\right)^2\right) \quad (1)$$

is approaching the cylinder from the left. U is the mean velocity in the channel of the undisturbed flow and L is the distance between the plates. Because the cylinder is placed halfway between the plates, the flow is assumed to be symmetric around the x -axis, for the considered Reynolds numbers. Furthermore the fluid is supposed to be incompressible, viscous and Newtonian.

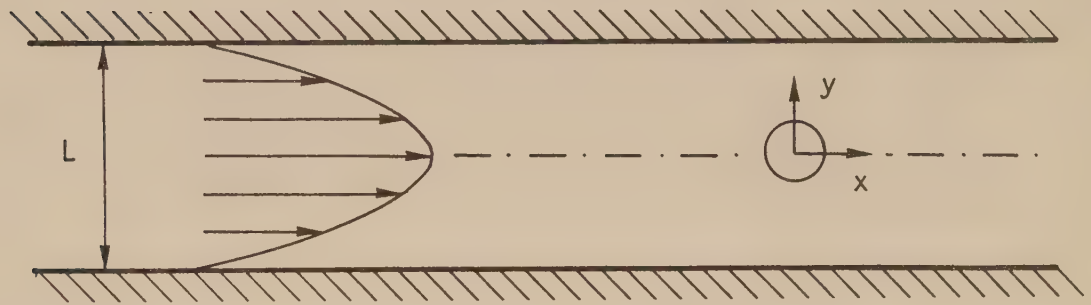


FIGURE 1. The flow field.

3 EQUATIONS

The flow is described by the equations of motion; the Navier-Stokes equation and the equation of continuity together with the boundary conditions.

The flow field is confined between boundaries of different shapes. In order to properly describe the boundary conditions for the arising difference equations the flow field is divided into two parts; the "inner" field and the "outer" field, which are overlapping in the "common" field; see fig 2. In the "inner" field close to the cylinder, the equations are expressed in circular cylindrical coordinates (r, θ) and in the "outer" field close to the channel wall and far from the cylinder they are expressed in rectangular coordinates (x, y) .

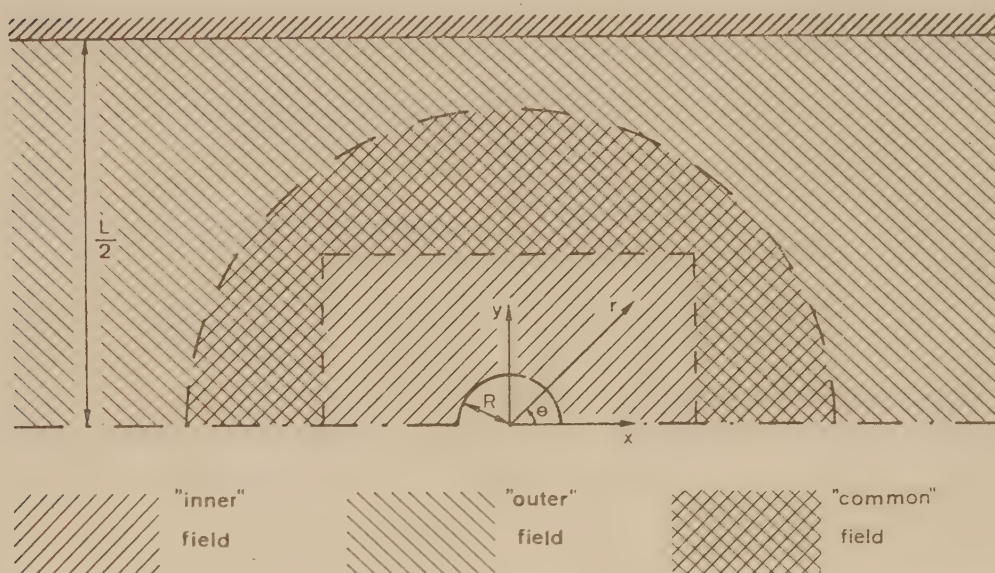
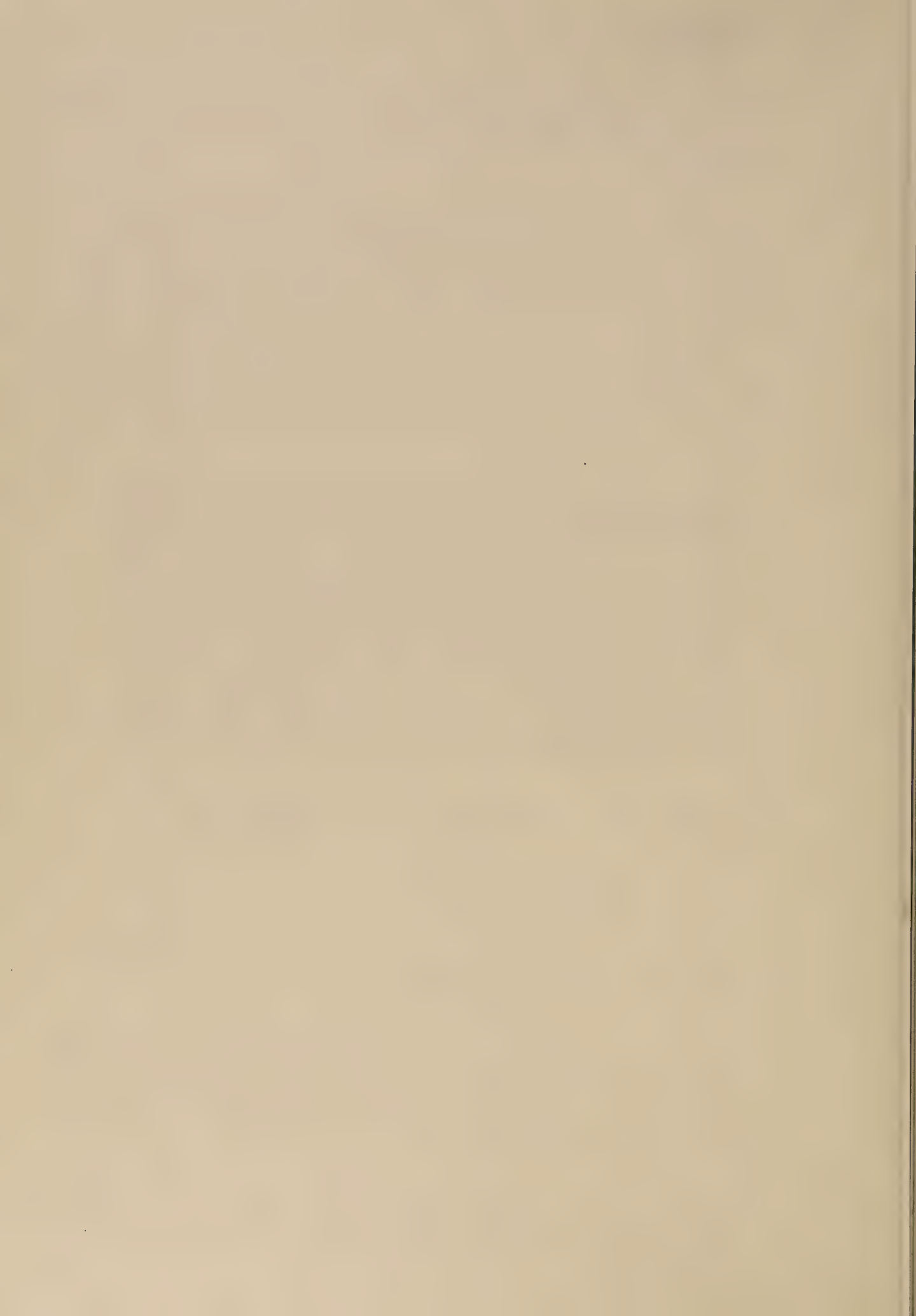


FIGURE 2. Computational fields.

In Appendix all basic equations are derived in order to be able to calculate streamlines, isovorticity lines, isobars etc.

In the following all equations are made dimensionless; lengths with respect to half the channel width, $L/2$, velocity components with respect to the undisturbed mean velocity, U , and pressure with respect to the dynamic head $\frac{1}{2} \rho U^2$.

Because of symmetry the calculations are carried out in only one half of the channel. The undisturbed flow is in positive x -direction (or $\theta = 0$).



4 STREAMLINES AND ISOVORTICITY LINES

4.1 BASIC EQUATIONS

4.1.1 "Inner" field

In the "inner" field the governing dimensionless equations for the streamfunction, ψ , and the vorticity, ω , in terms of the circular cylindrical coordinates (r, θ) become (A23, A24)

$$\nabla^2 \omega + \frac{\text{Re}}{2} \left[\frac{1}{r} \frac{\partial \omega}{\partial r} \frac{\partial \psi}{\partial \theta} - \frac{1}{r} \frac{\partial \omega}{\partial \theta} \frac{\partial \psi}{\partial r} \right] = 0 \quad (2)$$

$$\nabla^2 \psi - \omega = 0 \quad (3)$$

where ∇^2 is the Laplace operator

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \quad (4)$$

The Reynolds number, Re , is

$$\text{Re} = \frac{U \cdot L \cdot \rho}{\mu} \quad (5)$$

The velocity components, v_r , v_θ , are related to the streamfunction and the vorticity according to

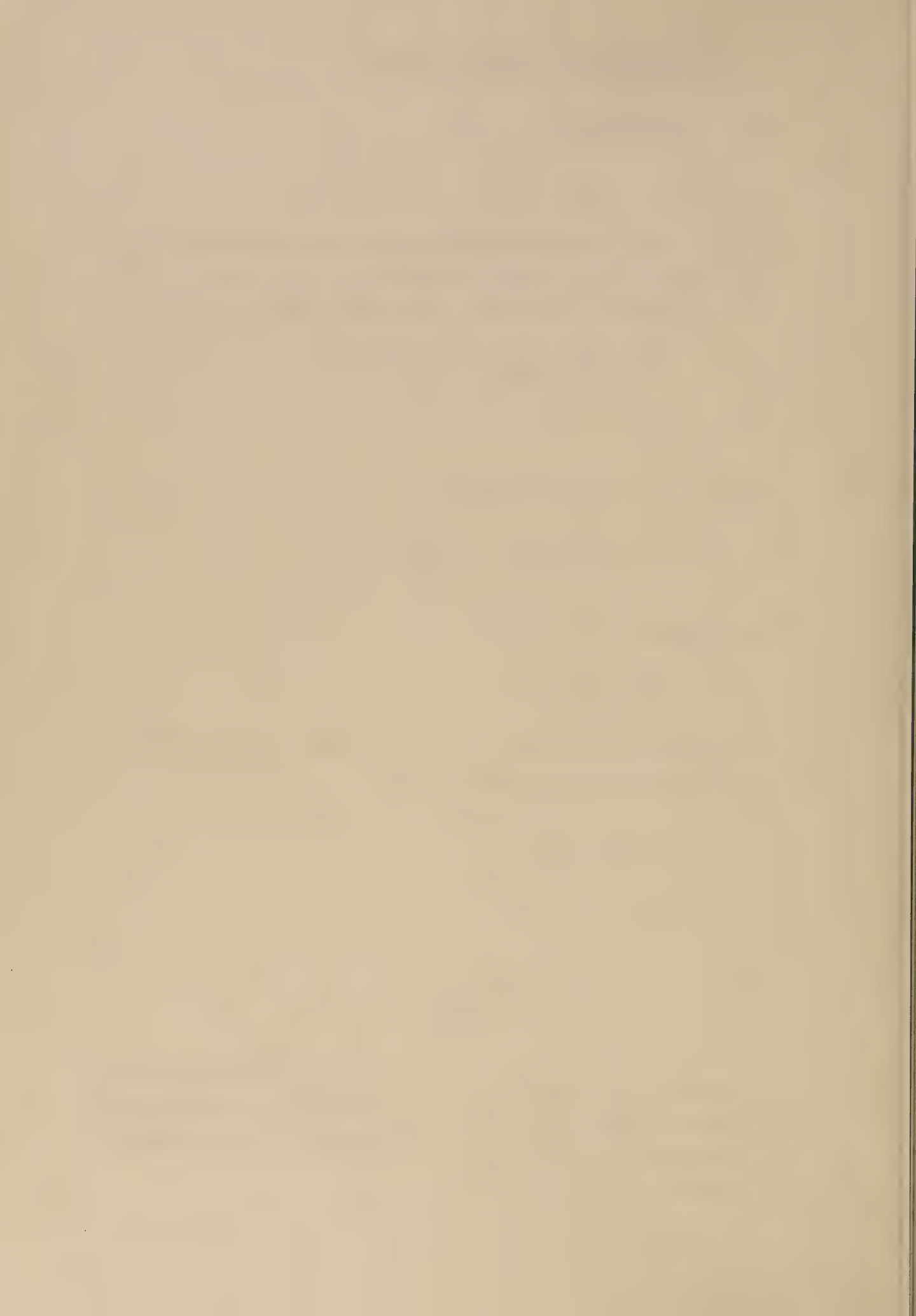
$$v_r = -\frac{1}{r} \frac{\partial \psi}{\partial \theta} \quad (6)$$

$$v_\theta = \frac{\partial \psi}{\partial r} \quad (7)$$

and

$$\omega = \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r} \frac{\partial v_r}{\partial \theta} \quad (8)$$

Since it is desirable to have a finer mesh close to the surface of the cylinder, where gradients are large, and to simplify the Laplace operator the following transformation of the independent variables (r, θ) is performed.



$$r = r_0 e^{\pi \xi} \quad (9)$$

$$\theta = \eta \pi \quad (10)$$

r_0 is the dimensionless cylinder radius.

Eq (2) and (3) in terms of the coordinates (ξ, η) now is

$$\nabla^2 \omega + \frac{\text{Re}}{2} \frac{1}{E^2} \left[\frac{\partial \omega}{\partial \xi} \frac{\partial \psi}{\partial \eta} - \frac{\partial \omega}{\partial \eta} \frac{\partial \psi}{\partial \xi} \right] = 0 \quad (11)$$

$$\nabla^2 \psi - \omega = 0 \quad (12)$$

where the operator, ∇^2 , is

$$\nabla^2 = \frac{1}{E^2} \left[\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right] \quad (13)$$

and E is

$$E = r_0 \pi e^{\pi \xi} \quad (14)$$

The relations between the streamfunction and the vorticity and the velocity components now read.

$$v_r = - \frac{1}{E} \frac{\partial \psi}{\partial \eta} \quad (15)$$

$$v_\theta = \frac{1}{E} \frac{\partial \psi}{\partial \xi} \quad (16)$$

$$\omega = \frac{1}{E} \left[\frac{\partial v_\theta}{\partial \xi} + \pi v_\theta - \frac{\partial v_r}{\partial \eta} \right] \quad (17)$$

The boundary conditions are

a) On the cylinder surface, $\xi = 0$

$$\left\{ \begin{array}{l} v_r = 0 \\ v_\theta = 0 \end{array} \right. \quad . \quad \left\{ \begin{array}{l} \psi = 0 \\ \frac{\partial \psi}{\partial \xi} = 0 \end{array} \right. \quad (18)$$



b) On the plane of symmetry, $\eta = 0$ and $\eta = 1$

$$\begin{cases} \frac{\partial v_r}{\partial \eta} = 0 \\ v_\theta = 0 \end{cases} \quad \begin{cases} \psi = 0 \\ \omega = 0 \end{cases} \quad (19)$$

c) On the "outer" boundary

$$\begin{cases} \psi = \psi_{\text{outer}} \\ \omega = \omega_{\text{outer}} \end{cases} \quad (20)$$

4.1.2 "Outer" field

The corresponding equations for the streamfunction, ψ , and the vorticity, ω , in rectangular coordinates (x,y) are (from eqs (A17) and (A18))

$$\nabla^2 \omega + \frac{\text{Re}}{2} \left[\frac{\partial \omega}{\partial x} \frac{\partial \psi}{\partial y} - \frac{\partial \omega}{\partial y} \frac{\partial \psi}{\partial x} \right] = 0 \quad (21)$$

$$\nabla^2 \psi - \omega = 0 \quad (22)$$

where the Laplace operator, ∇^2 , is

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad (23)$$

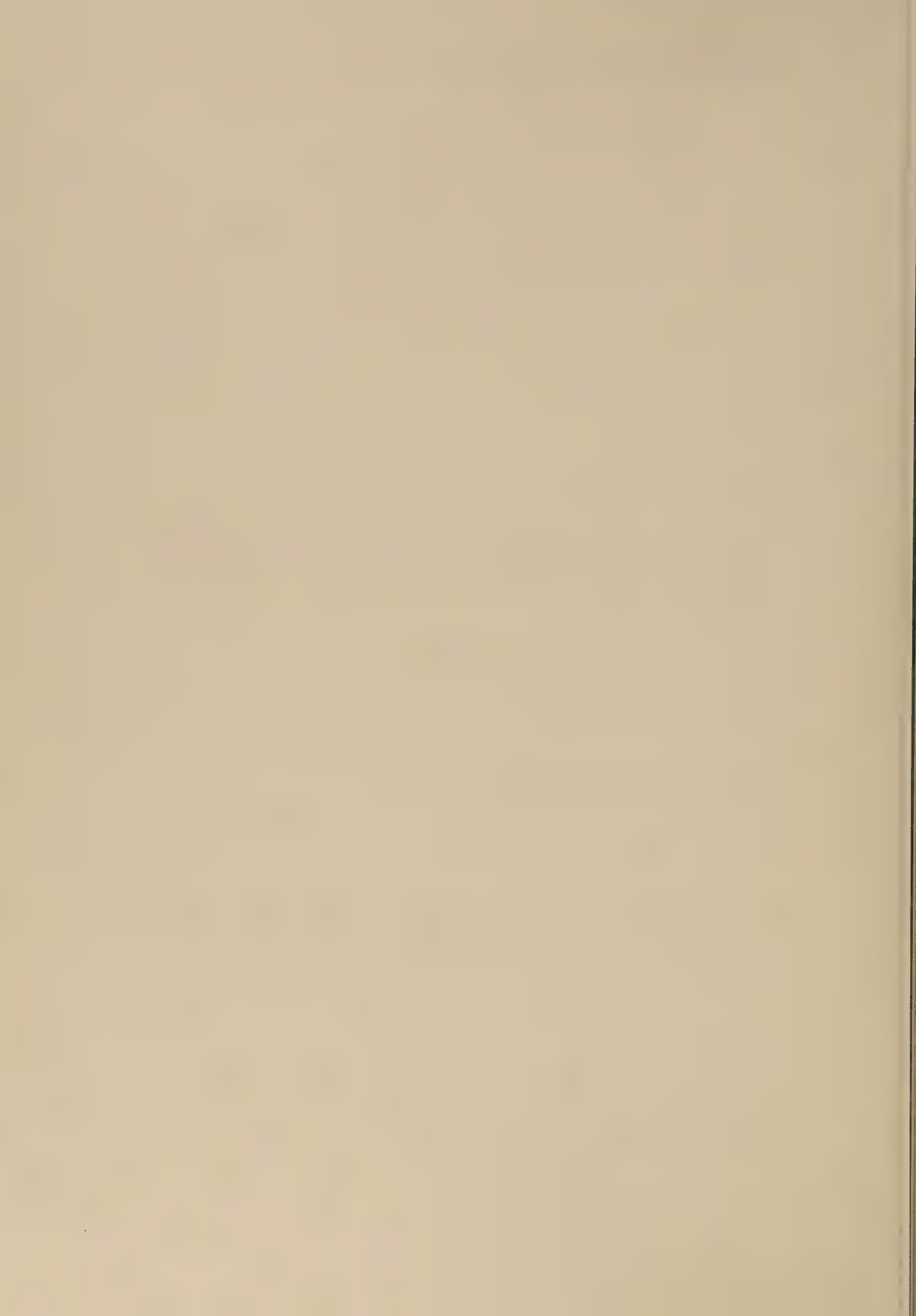
The Reynolds number, Re , is defined in eq (5). The velocity components are related to the streamfunction according to

$$v_x = - \frac{\partial \psi}{\partial y} \quad (24)$$

$$v_y = \frac{\partial \psi}{\partial x} \quad (25)$$

and to the vorticity

$$\omega = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \quad (26)$$



The boundary conditions for the "outer" field are

a) Far from the cylinder, $x \rightarrow \pm \infty$

$$\begin{cases} v_x = 3(1-y^2)/2 \\ v_y = 0 \end{cases} \quad \begin{cases} \psi = y(y^2-3)/2 \\ \omega = 3y \end{cases} \quad (27)$$

b) On the channel wall, $y = 1$

$$\begin{cases} v_x = 0 \\ v_y = 0 \end{cases} \quad \begin{cases} \psi = -1 \\ \frac{\partial \psi}{\partial y} = 0 \end{cases} \quad (28)$$

c) On the plane of symmetry, $y = 0$

$$\begin{cases} v_y = 0 \\ \frac{\partial v_x}{\partial y} = 0 \end{cases} \quad \begin{cases} \psi = 0 \\ \omega = 0 \end{cases} \quad (29)$$

d) On the "inner" boundary

$$\begin{cases} \psi = \psi_{\text{inner}} \\ \omega = \omega_{\text{inner}} \end{cases} \quad (30)$$



4.2 NUMERICAL METHODS

The governing differential equations for the flow field (11), (12), (21) and (22) with corresponding boundary conditions have been solved numerically. The differential equations have been transformed to difference equations. The method of solution is iterative and a relaxation procedure has been used.

4.2.1 "Inner" field

The differential equation (11) is written in difference form

$$\begin{aligned} \omega_{i,j}^{m+1} = & \rho_{\omega} \left\{ (\omega_{i-1,j}^{m+1} + \omega_{i+1,j}^m) / h_{\xi}^2 + (\omega_{i,j-1}^{m+1} + \omega_{i,j+1}^m) / h_{\eta}^2 + \right. \\ & + \frac{\text{Re}}{2} \left[(\omega_{i+1,j}^m - \omega_{i-1,j}^m) (\psi_{i,j+1}^{n+1} - \psi_{i,j-1}^{n+1}) - \right. \\ & \left. \left. - (\omega_{i,j+1}^m - \omega_{i,j-1}^m) (\psi_{i+1,j}^{n+1} - \psi_{i-1,j}^{n+1}) \right] / 4 h_{\xi} h_{\eta} \right\} \frac{h_{\xi}^2 h_{\eta}^2}{2(h_{\xi}^2 + h_{\eta}^2)} - \\ & - (\rho_{\omega} - 1) \omega_{i,j}^m \end{aligned} \quad (31)$$

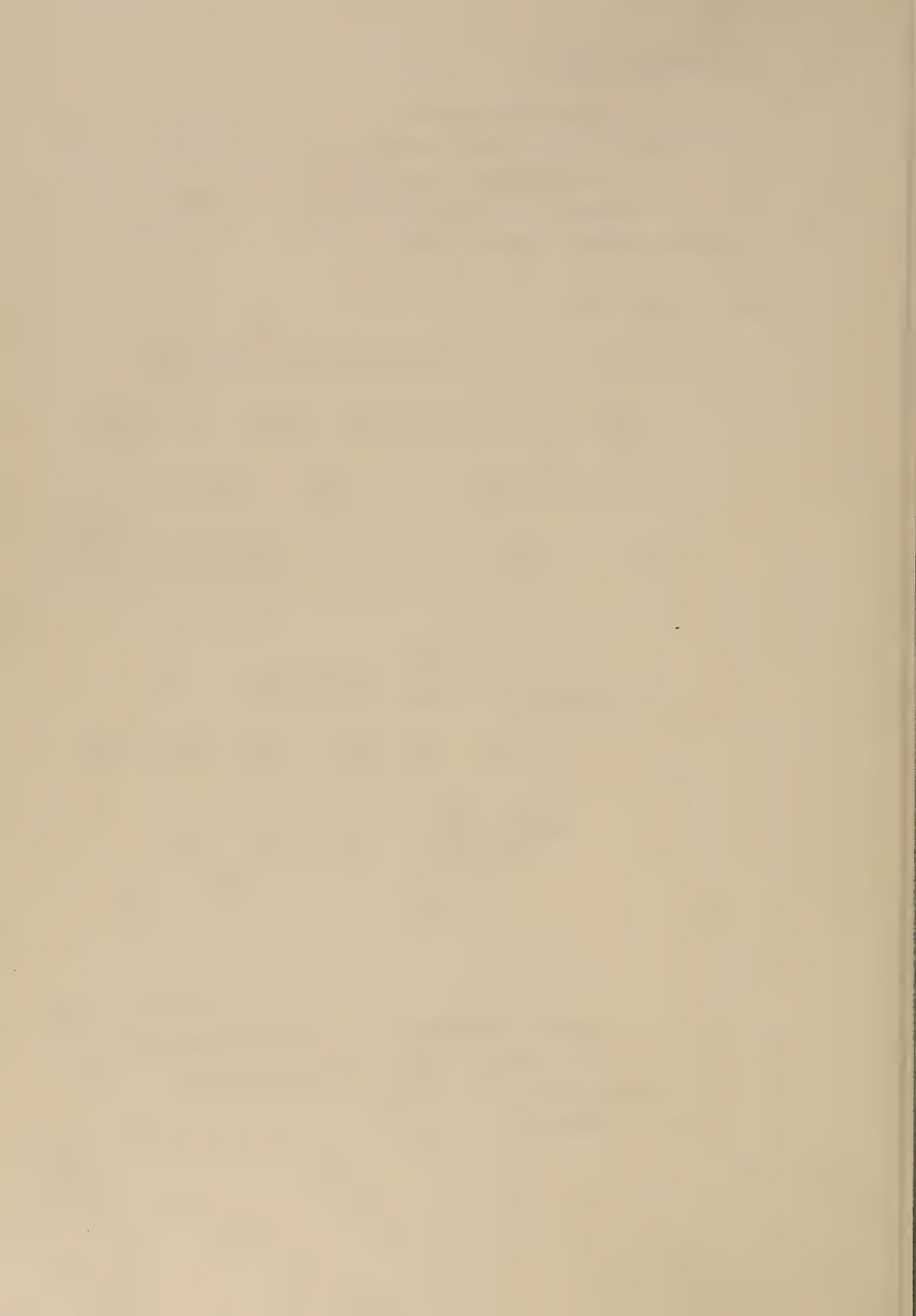
In a similar manner eq (12) can be expressed as

$$\begin{aligned} \psi_{i,j}^{n+1} = & \rho_{\psi} \left\{ (\psi_{i-1,j}^{n+1} + \psi_{i+1,j}^n) / h_{\xi}^2 + (\psi_{i,j-1}^{n+1} + \psi_{i,j+1}^n) / h_{\eta}^2 - \right. \\ & \left. - E^2 \omega_{i,j}^{m+1} \right\} \frac{h_{\xi}^2 h_{\eta}^2}{2(h_{\xi}^2 + h_{\eta}^2)} - (\rho_{\psi} - 1) \psi_{i,j}^n \end{aligned} \quad (32)$$

where

$$E = r_0 \pi e^{\pi(i-1)h_{\xi}} \quad (33)$$

The method consists of applying eqs (31) and (32) at every internal mesh point. The points are scanned from left to right with start in the left-hand bottom. One iteration is completed when all internal points are scanned.



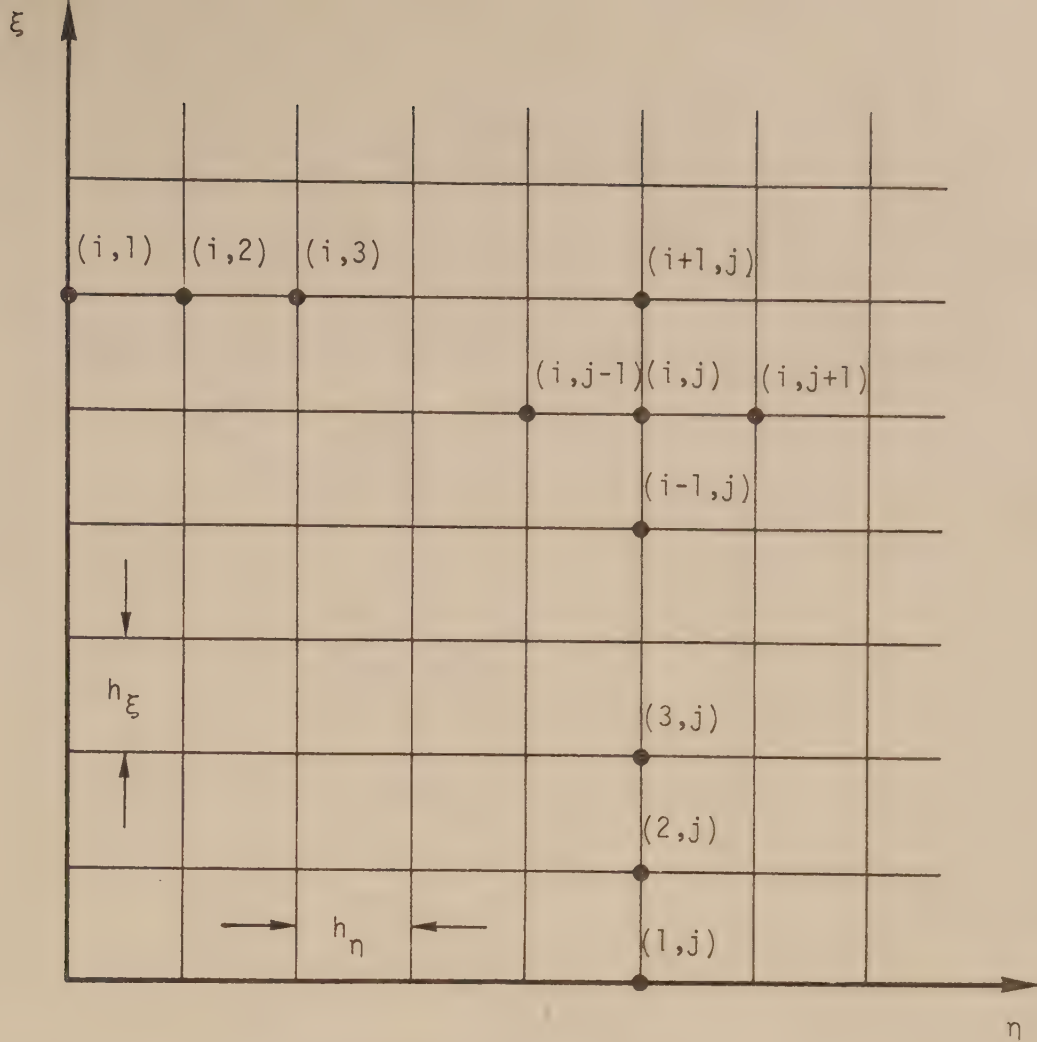


FIGURE 3. Grid structure, "inner" field.

$\omega_{i,j}^m$ and $\psi_{i,j}^n$ are approximations to the vorticity and the streamfunction respectively in the point (i,j) (corresponding to the coordinates $(\xi = (i-1)h_\xi, \eta = (j-1)h_\eta)$ at the m :th and n :th iteration respectively. ρ_ω and ρ_ψ are the relaxation factors.

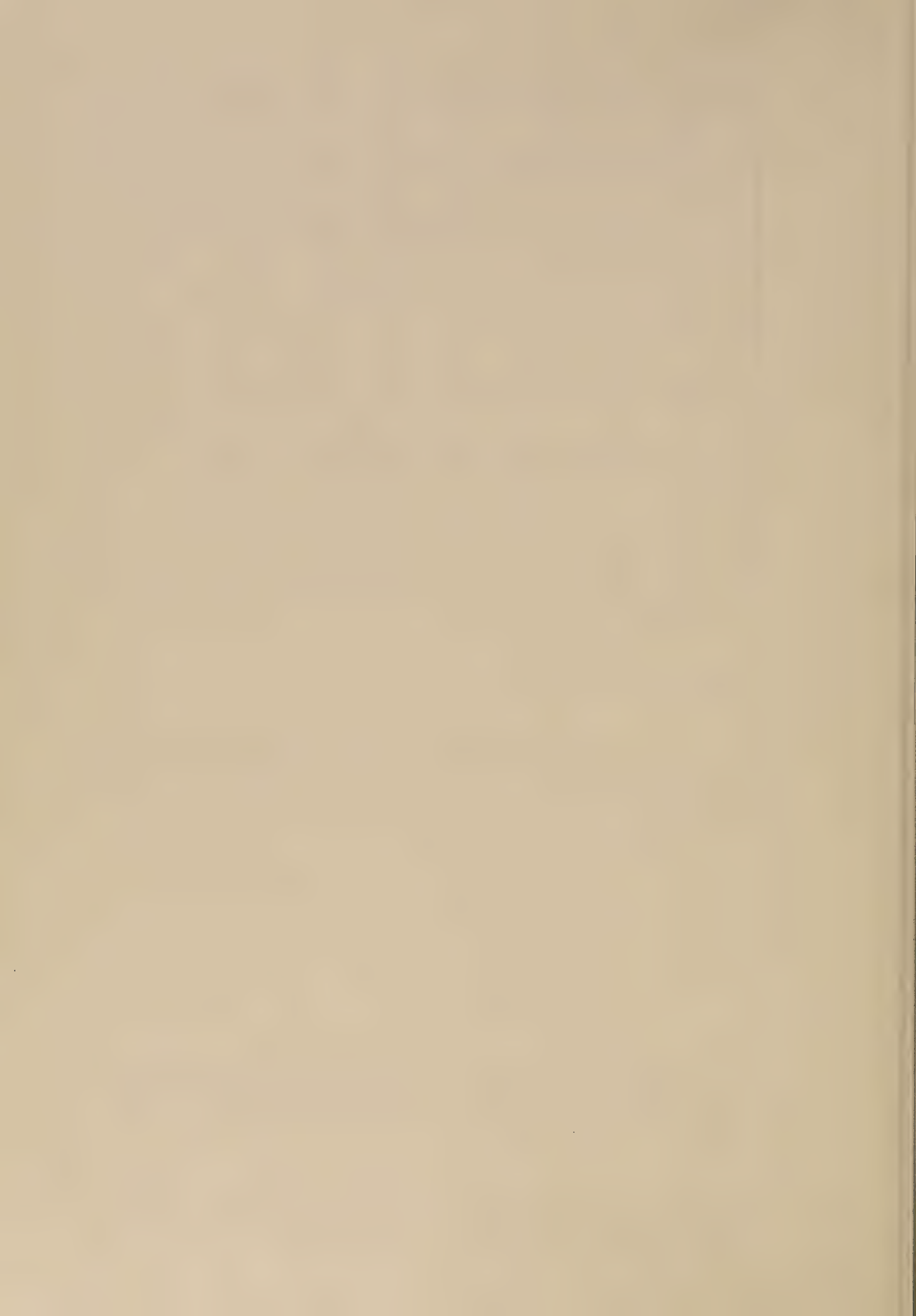
The numerical equations for the boundary conditions are

a) On the cylinder surface, $i = 1$

$$\psi_{1,j} = 0 \quad (34a)$$

The vorticity distribution on the surface of the cylinder has to be known. It is obtained from values of the streamfunction close to the cylinder.

The Taylor series expansion for the streamfunction at the points given by the coordinates $(2,j)$ and $(3,j)$ shown in fig. 3 are



$$\begin{cases} \psi_{2,j} = \psi_{1,j} + h_{\xi} \left(\frac{\partial \psi}{\partial \xi} \right)_{1,j} + \frac{h_{\xi}^2}{2} \left(\frac{\partial^2 \psi}{\partial \xi^2} \right)_{1,j} + \frac{h_{\xi}^3}{6} \left(\frac{\partial^3 \psi}{\partial \xi^3} \right)_{1,j} + \dots \\ \psi_{3,j} = \psi_{1,j} + 2h_{\xi} \left(\frac{\partial \psi}{\partial \xi} \right)_{1,j} + \frac{4h_{\xi}^2}{2} \left(\frac{\partial^2 \psi}{\partial \xi^2} \right)_{1,j} + \frac{8h_{\xi}^3}{6} \left(\frac{\partial^3 \psi}{\partial \xi^3} \right)_{1,j} + \dots \end{cases} \quad (35)$$

These two equations together with the boundary conditions

$$\psi_{1,j} = \left(\frac{\partial \psi}{\partial \xi} \right)_{1,j} = 0 \quad (34a, b)$$

lead to

$$\omega_{1,j} = \frac{8\psi_{2,j} - \psi_{3,j}}{2(\pi r_0 h_{\xi})^2} \quad (34c)$$

b) On the plane of symmetry, $j = 1$ and $j = J$

$$\begin{cases} \psi_{i,1} = \psi_{i,J} = 0 \\ \omega_{i,1} = \omega_{i,J} = 0 \end{cases} \quad (36)$$

c) On the "outer" boundary, $i = I$

$$\begin{cases} \psi_{I,j} = \psi_{\text{outer}} \\ \omega_{I,j} = \omega_{\text{outer}} \end{cases} \quad (37)$$

4.2.2 "Outer" field

The differential equations (21) and (22) can in difference form be expressed as

$$\begin{aligned} \omega_{k,l}^{m+1} = \rho_{\omega} \left\{ (\omega_{k-1,l}^{m+1} + \omega_{k+1,l}^m) / h_x^2 + (\omega_{k,l-1}^{m+1} + \omega_{k,l+1}^m) / h_y^2 + \right. \\ \left. + \frac{\text{Re}}{2} [(\omega_{k+1,l}^m - \omega_{k-1,l}^m)(\psi_{k,l+1}^{n+1} - \psi_{k,l-1}^{n+1}) - \right. \\ \left. - (\omega_{k,l+1}^m - \omega_{k,l-1}^m)(\psi_{k+1,l}^{n+1} - \psi_{k-1,l}^{n+1})] / 4h_x h_y \right\} \frac{h_x^2 h_y^2}{2(h_x^2 + h_y^2)} - \\ - (\rho_{\omega} - 1) \omega_{k,l}^m \end{aligned} \quad (38)$$



and

$$\begin{aligned} \psi_{k,l}^{n+1} = \rho_\psi \left\{ (\psi_{k-1,l}^{n+1} + \psi_{k+1,l}^n)/h_x^2 + (\psi_{k,l-1}^{n+1} + \psi_{k,l+1}^n)/h_y^2 - \right. \\ \left. - \omega_{k,l}^{m+1} \right\} \frac{h_x^2 h_y^2}{2(h_x^2 + h_y^2)} - (\rho_\psi - 1) \psi_{k,l}^n \end{aligned} \quad (39)$$

$\omega_{k,l}^m$ and $\psi_{k,l}^n$ are approximations of the vorticity and the stream-function at the point (k,l) , i.e. at the coordinates $(x = (k - k_{cyl})h_x, y = (l - 1)h_y)$. $(k_{cyl}, 1)$ is the mesh point at the centre of the cylinder.

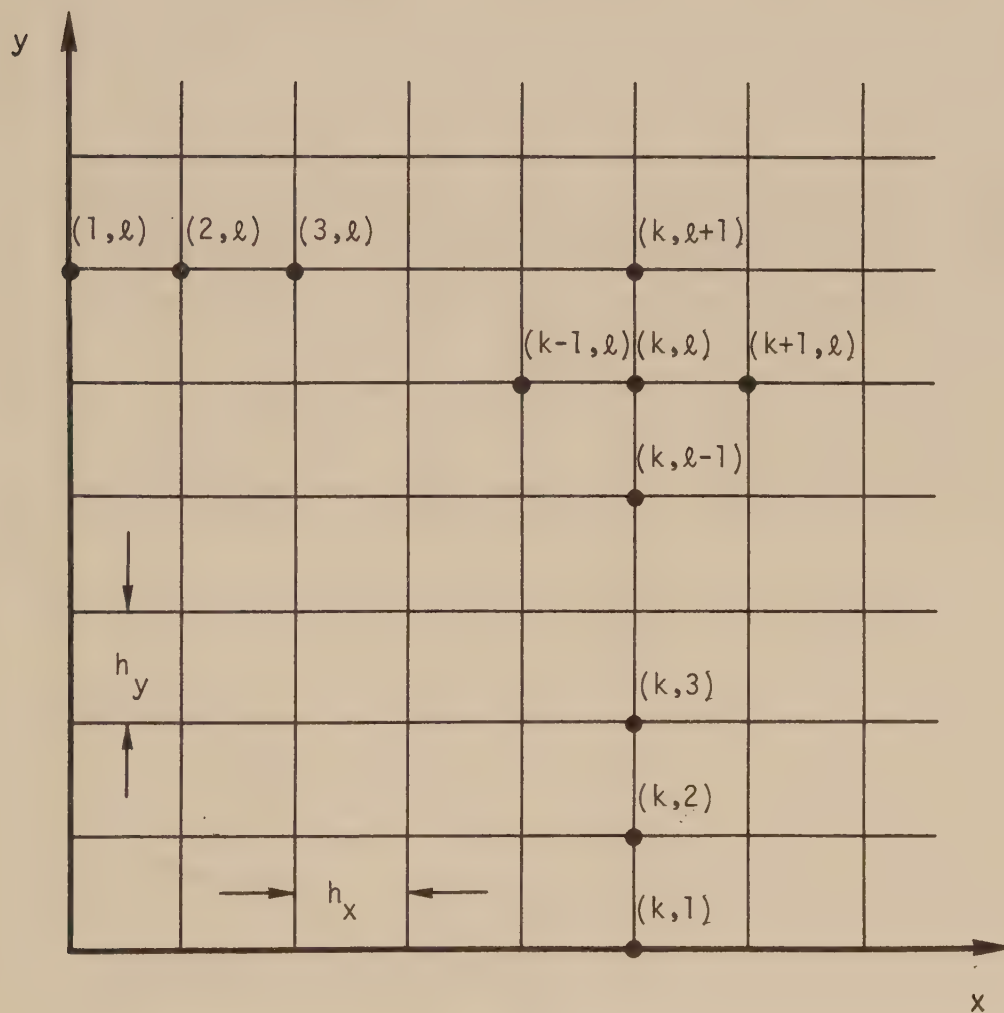


FIGURE 4. Grid structure, "outer" field.

The internal mesh points are scanned row by row in a manner similar to that for the "inner" field. m and n are also here the number of iterations, and ρ_ω and ρ_ψ the relaxation factors.



The boundary conditions for the "outer" field in numerical form are

a) Far from the cylinder, $k = 1$ and $k = K$

$$\begin{cases} \psi_{1,\ell} = \psi_{K,\ell} = y(y^2 - 3)/2 \\ \omega_{1,\ell} = \omega_{K,\ell} = 3y \end{cases} \quad (40)$$

where $y = (\ell - 1)h_y$

b) On the channel wall, $\ell = L$

$$\psi_{k,L} = -1 \quad (41a)$$

The vorticity distribution on the channel wall must be calculated. It is derived in the similar manner as the vorticity distribution on the surface of the cylinder. The resulting expression is

$$\omega_{k,L} = (8\psi_{k,L-1} - \psi_{k,L-2} + 7)/2h_y^2 \quad (41b)$$

where the boundary conditions

$$\psi_{k,L} = -1 \quad (41a)$$

and

$$\left(\frac{\partial \psi}{\partial y}\right)_{k,L} = 0 \quad (41c)$$

have been used.

c) On the plane of symmetry, $\ell = 1$

$$\begin{cases} \psi_{k,1} = 0 \\ \omega_{k,1} = 0 \end{cases} \quad (42)$$

d) On the "inner" boundary

$$\psi_{k,\ell} = \psi_{\text{inner}} \quad (43)$$



4.3 CALCULATION

Initial values of ψ and ω are obtained by supposing undisturbed Poiseuille flow together with the appropriate boundary conditions. The vorticity distribution on the surface of the cylinder is assumed to be zero. (This procedure is performed only for $Re = 0$. When a solution is obtained for $Re = 0$, the calculated values of ψ and ω are used as initial values for calculation of $Re = 1$ and so on.)

The calculation starts with iteratively solving eq (31) until the convergence criterion

$$|\omega_{i,j}^{m+1} - \omega_{i,j}^m| < \varepsilon_\omega |\omega_{i,j}^m| \quad (44)$$

is satisfied for all internal points. The obtained values of $\omega_{i,j}^{m+1}$ are then used in eq (32) to calculate $\psi_{i,j}^{n+1}$. The convergence criterion between consecutive iterations should be satisfied

$$|\psi_{i,j}^{n+1} - \psi_{i,j}^n| < \varepsilon_\psi |\psi_{i,j}^n| \quad (45)$$

The vorticity distribution on the surface of the cylinder is now calculated with eq (34c). If the condition

$$|\omega_{1,j}^{new} - \omega_{1,j}^{old}| < \varepsilon_\omega |\omega_{1,j}^{old}| \quad (46)$$

is not satisfied, the computation restarts with solving eq (31) with the new values of ω at $i = 1$.

When the convergence criterions (44), (45) and (46) are satisfied for only one iteration of (31) and (32) respectively at a consecutive calculation, the solution is accepted.

The accepted solution for the "inner" field is now used to calculate the values ω_{inner} and ψ_{inner} on the "inner" boundary of the "outer" field. The calculation of the "outer" field begins with eq (38), with the new values of ω_{inner} on the "inner" boundary. The equation is iterated until the convergence criterion

$$|\omega_{k,l}^{m+1} - \omega_{k,l}^m| < \varepsilon_\omega |\omega_{k,l}^m| \quad (47)$$



is satisfied for all internal points.

Eq (39) is then solved with the accepted values of $\omega_{k,l}^{m+1}$ and the appropriate boundary conditions. The solution of the streamfunction is accepted when the condition

$$|\psi_{k,l}^{n+1} - \psi_{k,l}^n| < \varepsilon_\psi |\psi_{k,l}^n| \quad (48)$$

is satisfied.

The vorticity distribution on the channel wall is now calculated with eq (41b). If the condition

$$|\omega_{k,L}^{\text{new}} - \omega_{k,L}^{\text{old}}| < \varepsilon_\omega |\omega_{k,L}^{\text{old}}| \quad (49)$$

is not satisfied, eq (38) is recalculated with the new values of $\omega_{k,L}$. The calculation procedure is then repeated until the convergence criterions (47), (48) and (49) are satisfied for only one iteration of (38) and (39) at a consecutive calculation. The solution for the "outer" field is then accepted.

New values of ω_{outer} and ψ_{outer} are calculated on the "outer" boundary of the "inner" field.

If the convergence criterions

$$\begin{cases} |\omega^{\text{new}} - \omega^{\text{old}}| < \varepsilon_\omega |\omega^{\text{old}}| \\ |\psi^{\text{new}} - \psi^{\text{old}}| < \varepsilon_\psi |\psi^{\text{old}}| \end{cases} \quad (50)$$

on the "inner" boundary for the "outer" field and on the "outer" boundary for the "inner" field are not satisfied, the calculation procedure is restarted.

If the conditions (50) are satisfied and there has been only one iteration of each of the four eqs (31), (32), (38) and (39) in a consecutive calculation, the solution of the "inner" and the "outer" fields is accepted. If the iteration criterion is not satisfied the computation begins again with eq (31).

For flow sheet, see Appendix 2.



5 ISOBARS

5.1 BASIC EQUATIONS

The basic equations for the pressure calculations are derived in Appendix 1.

5.1.1 "Inner" field

The pressure for the "inner" field can be calculated from eq (A32), which in dimensionless form is

$$\nabla^2 p + 4 \left[\left(\frac{\partial v_r}{\partial r} \right)^2 + \frac{1}{r} \frac{\partial v_\theta}{\partial r} \left(\frac{\partial v_r}{\partial \theta} - v_\theta \right) \right] = 0 \quad (51)$$

where ∇^2 is the Laplace operator (4).

The independent variables are transformed according to the above, eqs (9) and (10). Eq (51) now is

$$\nabla^2 p + \frac{4}{E^2} \left[\left(\frac{\partial v_r}{\partial \xi} \right)^2 + \frac{\partial v_\theta}{\partial \xi} \left(\frac{\partial v_r}{\partial \eta} - \pi v_\theta \right) \right] = 0 \quad (52)$$

where the Laplace operator is

$$\nabla^2 = \frac{1}{E^2} \left[\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} \right] \quad (13)$$

and E is

$$E = r_0 \pi e^{\pi \xi} \quad (14)$$

The velocity components are related to the streamfunction according to eqs (15) and (16).

The pressure distribution on the boundaries has to be calculated. The dimensionless Navier-Stokes equation in form of eq (A33) is used. The pressure distribution on the surface of the cylinder is calculated from the θ -component of eq (A33), which in dimensionless form together with the boundary conditions $v_r = v_\theta = 0$ becomes

$$\frac{1}{r} \frac{\partial p}{\partial \theta} = \frac{4}{Re} \frac{\partial \omega}{\partial r} \quad (53)$$

After the coordinate transformation is performed eq (53) reads

$$\frac{\partial p}{\partial \eta} = \frac{4}{\text{Re}} \frac{\partial \omega}{\partial \xi} \quad (54)$$

In a similar manner the pressure distribution on the plane of symmetry is obtained from the r-component of eq (A33)

$$\frac{\partial p}{\partial r} = - \frac{4}{\text{Re}} \frac{1}{r} \frac{\partial \omega}{\partial \theta} - \frac{\partial v^2}{\partial r} \quad (55)$$

Where the boundary condition $\omega = 0$ has been used.

The coordinate transformation gives

$$\frac{\partial p}{\partial \xi} = - \frac{4}{\text{Re}} \frac{\partial \omega}{\partial \eta} - \frac{\partial v^2}{\partial \xi} \quad (56)$$

The pressure on the cylinder surface at $\theta = \pi/2$ is taken as a reference pressure in the pressure calculations, i.e.

$$p_{\text{ref}} = p(r_0, \pi/2) \quad (57)$$

The boundary conditions for the "inner" field now become

a) On the surface of the cylinder, $\xi = 0$

$$p = \frac{4}{\text{Re}} \int_{1/2}^{\eta} \frac{\partial \omega}{\partial \xi} d\eta + p_{\text{ref}} \quad (58)$$

b) On the plane of symmetry, $\eta = 0$ and $\eta = 1$

$$p = - \frac{4}{\text{Re}} \int_0^{\xi} \frac{\partial \omega}{\partial \eta} d\xi - v_{\xi=\xi}^2 + p_{\xi=0} \quad (59)$$

because $v_{\xi=0}^2 = 0$.

c) On the "outer" boundary

$$p = p_{\text{outer}} \quad (60)$$



5.1.2 "Outer" field

The "pressure equation" for the "outer" field is derived in eq (A32), which in dimensionless form becomes

$$\nabla^2 p + 4 \left[\left(\frac{\partial v_x}{\partial x} \right)^2 + \frac{\partial v_x}{\partial y} \frac{\partial v_y}{\partial x} \right] = 0 \quad (61)$$

where ∇^2 is the Laplace operator (23). The velocity components are related to the streamfunction according to eqs (24) and (25).

The pressure distribution is calculated in a manner similar to that for the "inner" field. The x-component of eq (A33) is used. The pressure distribution on the wall is calculated from

$$\frac{\partial p}{\partial x} = - \frac{4}{Re} \frac{\partial \omega}{\partial y} \quad (62)$$

where the boundary condition $v_x = v_y = 0$ has been used.

The pressure distribution on the plane of symmetry is described by

$$\frac{\partial p}{\partial x} = - \frac{4}{Re} \frac{\partial \omega}{\partial y} - \frac{\partial v^2}{\partial x} \quad (63)$$

because $\omega = 0$ for $y = 0$.

The boundary conditions are

a) On the plane of symmetry, $y = 0$

$$p = - \frac{4}{Re} \int_{x_{\text{common}}}^x \frac{\partial \omega}{\partial y} dx - (v_{x=x}^2 - v_{x=x_{\text{common}}}^2) + p_{x=x_{\text{common}}} \quad (64)$$

where x_{common} is a point in the "common" field for $y = 0$.

b) Far from the cylinder, $x \rightarrow \pm\infty$

$$\begin{cases} p = p_{+\infty} \\ p = p_{-\infty} \end{cases} \quad (65)$$



c) On the channel wall, $y = 1$

$$p = - \frac{4}{\text{Re}} \int_{-\infty}^x \frac{\partial \omega}{\partial y} dx + p_{-\infty} \quad (66)$$

d) On the "inner" boundary

$$p = p_{\text{inner}}$$



5.2 NUMERICAL METHODS

The method of solution of the pressure is similar to that of the streamfunction and the vorticity.

5.2.1 "Inner" field

The differential equation (52) is transformed to the difference equation

$$p_{i,j}^{m+1} = \rho_p \left\{ (p_{i-1,j}^{m+1} + p_{i+1,j}^m)/h_\xi^2 + (p_{i,j-1}^{m+1} + p_{i,j+1}^m)/h_\eta^2 + \right. \\ \left. + \phi_{i,j} \right\} \frac{h_\xi^2 h_\eta^2}{2(h_\xi^2 + h_\eta^2)} - (\rho_p - 1)p_{i,j}^m \quad (67)$$

$\phi_{i,j}$ is calculated from the solution of the streamfunction

$$\phi_{i,j} = 4 \left\{ (v_{r_{i+1,j}} - v_{r_{i-1,j}})^2 / 4h_\xi^2 + (v_{\theta_{i+1,j}} - \right. \\ \left. - v_{\theta_{i-1,j}})(v_{r_{i,j+1}} - v_{r_{i,j-1}} - 2h_\eta \pi v_{\theta_{i,j}}) / 4h_\xi h_\eta \right\} \quad (68)$$

The velocity components are expressed as

$$v_{r_{i,j}} = -\frac{1}{E} (\psi_{i,j+1} - \psi_{i,j-1}) / 2h_\eta \quad (69)$$

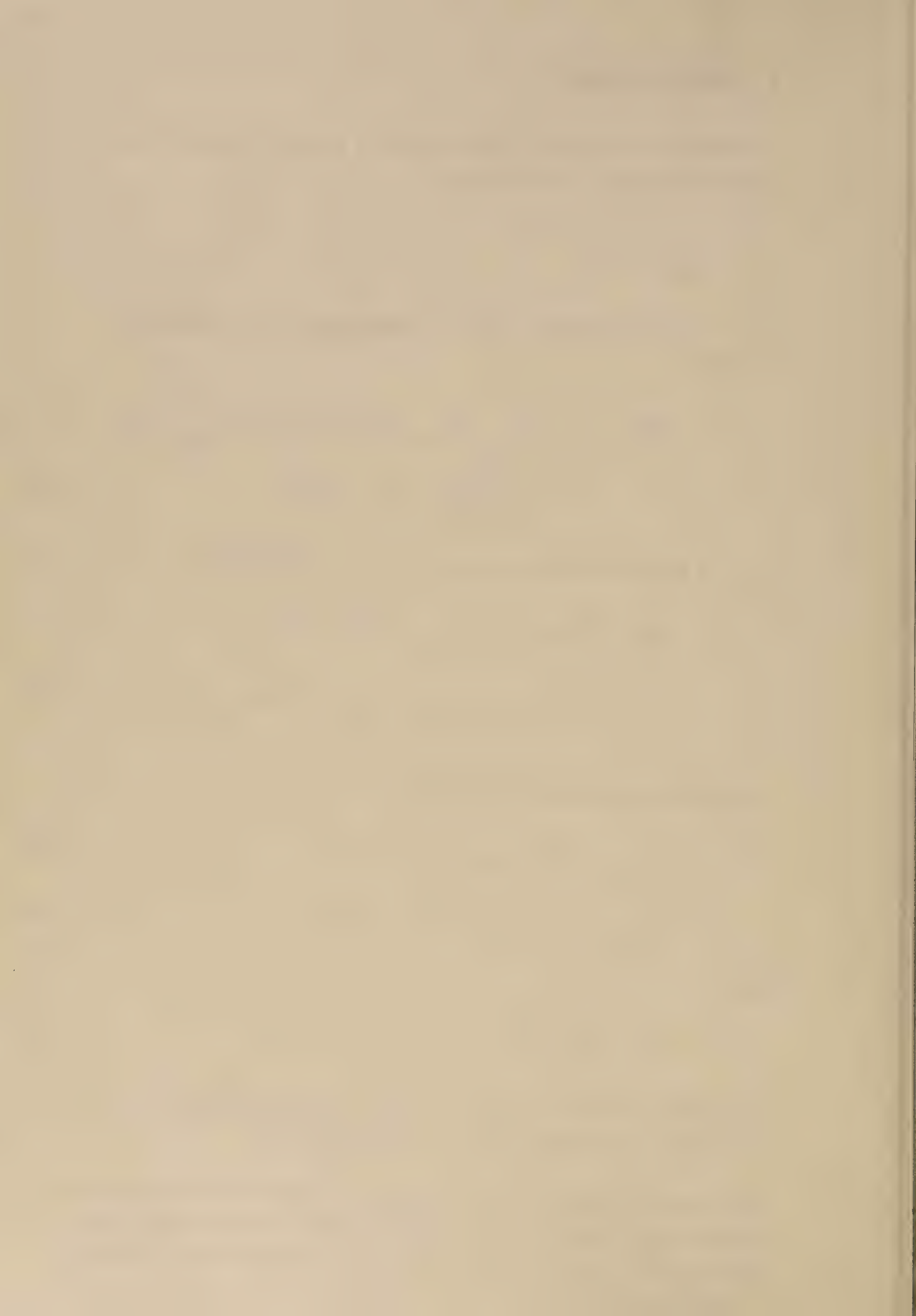
$$v_{\theta_{i,j}} = \frac{1}{E} (\psi_{i+1,j} - \psi_{i-1,j}) / 2h_\xi \quad (70)$$

where

$$E = r_0 \pi e^{\pi(i-1)h_\xi} \quad (33)$$

The mesh is defined in fig 3. The iterations are performed equivalent to those for the vorticity and the streamfunction.

The boundary conditions are also written in numerical form. The derivatives $(\frac{\partial \omega}{\partial \xi}, \frac{\partial \omega}{\partial \eta})$ etc) involved in the equations are calculated with a Taylors series expansion. In the following they are not written out in their expanded form.



The boundary conditions are transformed to a numerical form.

a) On the surface of the cylinder, $i = 1$

The pressure $p_{1,j}$ is expressed implicit in the algorithm

$$p_{1,j} = p_{1,j-1} + \frac{4}{\text{Re}} \left[\left(\frac{\partial \omega}{\partial \xi} \right)_{1,j-1} + \left(\frac{\partial \omega}{\partial \xi} \right)_{1,j} \right] \frac{h_\eta}{2} \quad ; j = 2, J \quad (71a)$$

where

$$\begin{cases} p_{1,(J+1)/2} = p_{\text{ref}} & \text{if } J \text{ odd} \\ (p_{1,J/2} + p_{1,J/2+1})/2 + \frac{4}{\text{Re}} \left[\left(\frac{\partial \omega}{\partial \xi} \right)_{1,J/2} - \left(\frac{\partial \omega}{\partial \xi} \right)_{1,J/2+1} \right] \frac{h_\eta}{8} = p_{\text{ref}} & \text{if } J \text{ even} \end{cases} \quad (71b)$$

b) On the plane of symmetry, $j = 1$ and $j = J$

$$\begin{aligned} p_{i,j} = p_{i-1,j} - \frac{4}{\text{Re}} \left[\left(\frac{\partial \omega}{\partial \eta} \right)_{i-1,j} + \left(\frac{\partial \omega}{\partial \eta} \right)_{i,j} \right] \frac{h_\xi}{2} - \\ - (v_{i,j}^2 - v_{i-1,j}^2) \quad ; i = 2, I \end{aligned} \quad (72a)$$

where

$$v_{i,j}^2 = \frac{1}{E^2} \left(\frac{\partial \psi}{\partial \eta} \right)_{i,j}^2 \quad (72b)$$

and $p_{1,j}$ is calculated from boundary condition a) for the "inner" field.

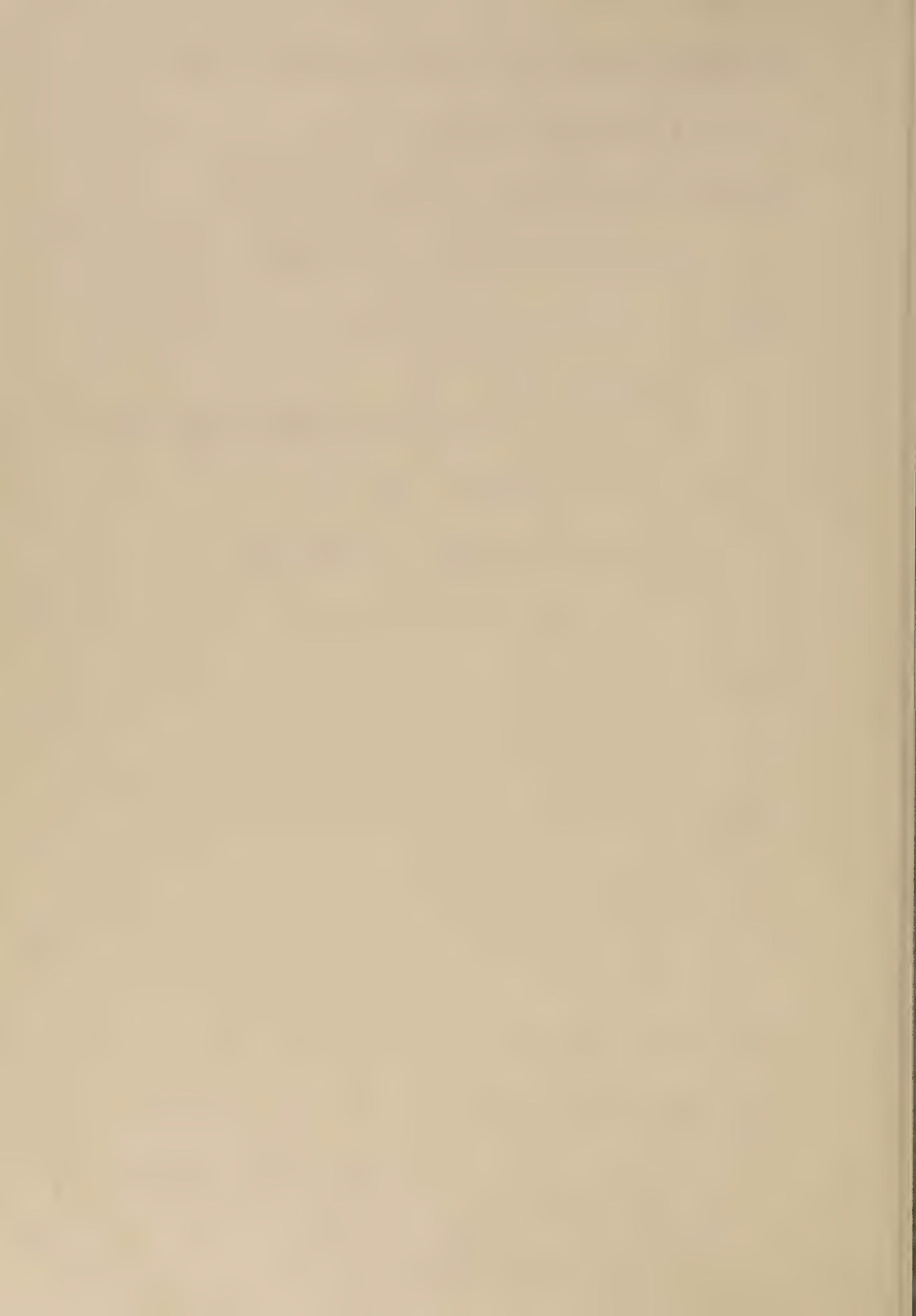
c) On the "outer" boundary, $i = I$

$$p_{I,j} = p_{\text{outer}} \quad (73)$$

5.2.2 "Outer" field

The difference form of eq (61) is

$$\begin{aligned} p_{k,l}^{n+1} = \rho_p \left\{ (p_{k-1,l}^{n+1} + p_{k+1,l}^n)/h_x^2 + (p_{k,l+1}^{n+1} + p_{k,l+1}^n)/h_y^2 + \right. \\ \left. + \phi_{k,l} \right\} \frac{h_x^2 h_y^2}{2(h_x^2 + h_y^2)} - (\rho_p - 1) p_{k,l}^n \end{aligned} \quad (74)$$



$\Phi_{k,l}$ is obtained from the calculated values of $\psi_{k,l}$

$$\Phi_{k,l} = 4 \left\{ (v_{x_{k+1,l}} - v_{x_{k-1,l}})^2 / 4h_x^2 + \right. \\ \left. + (v_{x_{k,l+1}} - v_{x_{k,l-1}})(v_{y_{k+1,l}} - v_{y_{k-1,l}}) / 4h_x h_y \right\} \quad (75)$$

The velocity components are related to the streamfunction according to

$$v_{x_{k,l}} = - (\psi_{k,l+1} - \psi_{k,l-1}) / 2h_y \quad (76)$$

$$v_{y_{k,l}} = (\psi_{k+1,l} - \psi_{k-1,l}) / 2h_x \quad (77)$$

For definition of grid and iteration performance, see the corresponding calculations of the vorticity and the streamfunction for the "outer" field.

The boundary conditions become

a) On the plane of symmetry, $l = 1$

$$p_{k,1} = p_{k-1,1} - \frac{4}{Re} \left[\left(\frac{\partial \omega}{\partial y} \right)_{k-1,1} + \left(\frac{\partial \omega}{\partial y} \right)_{k,1} \right] \frac{h_x}{2} - \\ - (v_{k,1}^2 - v_{k-1,1}^2) \quad (78a)$$

where

$$v_{k,1}^2 = \left(\frac{\partial \psi}{\partial y} \right)_{k,1}^2 \quad (78b)$$

and

$$\begin{cases} p_{k,1} = p_{inner(j=J)} & ; k = 2, k_{inner(j=J)} \\ p_{k-1,1} = p_{inner(j=1)} & ; k = k_{inner(j=1)} + 1, K \end{cases} \quad (78c)$$

$p_{inner(j=J)}$ and $p_{inner(j=1)}$ are the pressures at the "inner" boundary for the "outer" field in front of and behind the cylinder respectively.



b) Far from the cylinder, $k = 1$ and $k = K$

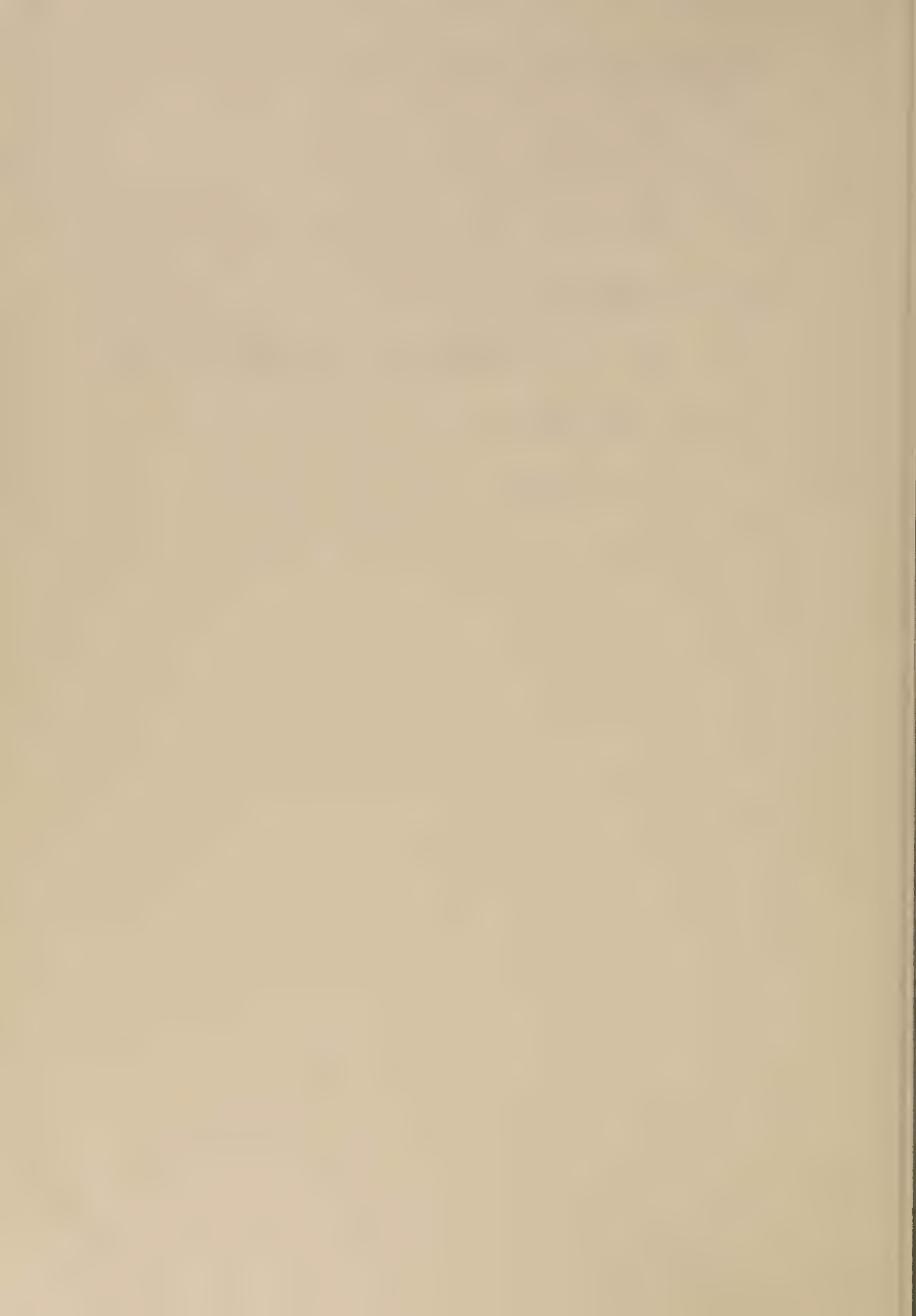
$$\begin{cases} p_{1,\ell} = p_{1,1} \\ p_{K,\ell} = p_{K,1} \end{cases} \quad (79)$$

c) On the channel wall, $\ell = L$

$$p_{k,L} = p_{k-1,L} - \frac{4}{\text{Re}} \left[\left(\frac{\partial \omega}{\partial y} \right)_{k-1,L} + \left(\frac{\partial \omega}{\partial y} \right)_{k,L} \right] \frac{h_x}{2} \quad ; \quad k = 2, K \quad (80)$$

d) On the "inner" boundary

$$p_{k,\ell} = p_{\text{inner}} \quad (81)$$



5.3 CALCULATION

The boundary conditions and the function ψ are calculated from the earlier obtained values of the vorticity and the streamfunction. Initial values of p are obtained by assuming a linear pressure drop for both the "inner" and the "outer" fields. Equation (67) is then iteratively computed to such an extent that the convergence criterion

$$|p_{i,j}^{m+1} - p_{i,j}^m| < \epsilon_p |p_{i,j}^m| \quad (82)$$

is satisfied.

Values of p_{inner} on the "inner" boundary of the "outer" field is now calculated. Eq (74) is then solved and the solution is accepted if the condition

$$|p_{k,l}^{n+1} - p_{k,l}^n| < \epsilon_p |p_{k,l}^n| \quad (83)$$

is satisfied.

New values of p_{outer} on the "outer" boundary of the "inner" field is calculated.

If the convergence criterions

$$|p_{inner}^{new} - p_{inner}^{old}| < \epsilon_p |p_{inner}^{old}| \quad (84a)$$

and

$$|p_{outer}^{new} - p_{outer}^{old}| < \epsilon_p |p_{outer}^{old}| \quad (84b)$$

are not satisfied, the calculation procedure is restarted.

If eqs (84a and b) are satisfied, and there has been only one iteration of eqs (67) and (74) at a consecutive calculation, the solution is accepted. If the number of iterations exceeds one for either eq (67) or (74) or for both, the computation is restarted with eq (67).

The flow sheet for the pressure calculations is presented in Appendix 3.



6 DRAG ON THE CYLINDER, PRESSURE DROP AND FORCE ON THE WALLS

The drag per unit length of the cylinder can be expressed in terms of a dimensionless drag coefficient, C_D

$$C_D = F_D / (2R \frac{\rho U^2}{2}) \quad (85)$$

where F_D is the total drag per unit length on the cylinder. R is the radius of the cylinder.

The drag coefficient is composed of two parts, one due to friction drag and the other to pressure drag. The pressure drag coefficient, C_{Dp} , is calculated from eq (A46).

$$C_{Dp} = - \frac{1}{2} \int_0^{2\pi} p_{r=r_0} \cos \theta \, d\theta \quad (86)$$

Because of symmetry eq (86) can be expressed as

$$C_{Dp} = - \pi \int_0^1 p_{\xi=0} \cos(\eta \cdot \pi) \, d\eta \quad (87)$$

where also the coordinate transformation is performed.

The friction drag coefficient, C_{Df} , is calculated from eq (A47)

$$C_{Df} = - \frac{2}{Re} \int_0^{2\pi} \omega_{r=r_0} \sin \theta \, d\theta \quad (88)$$

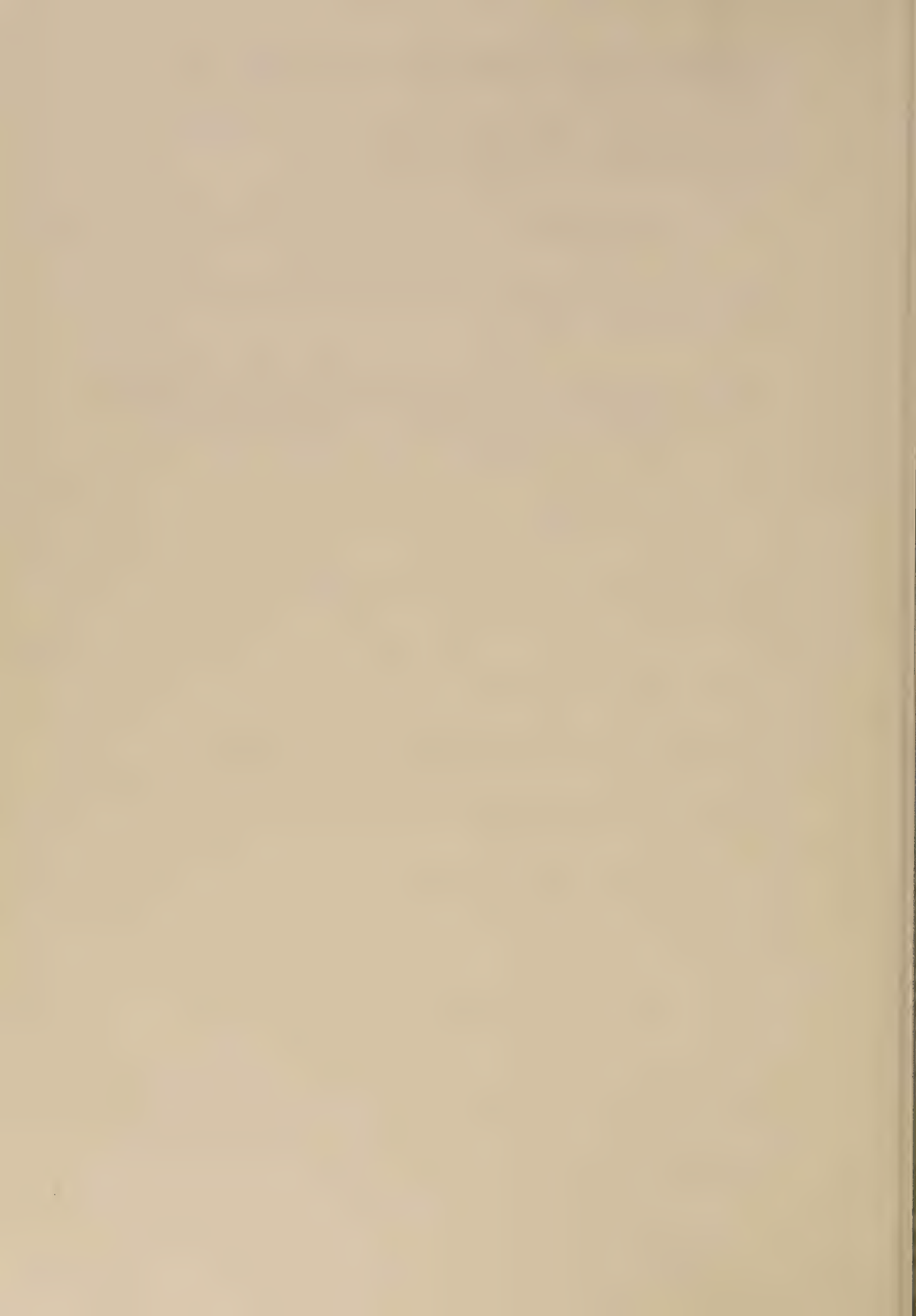
or

$$C_{Df} = - \frac{4}{Re} \int_0^1 \omega_{\xi=0} \sin(\eta \cdot \pi) \, d\eta \quad (89)$$

where the symmetry condition is used.

The drag coefficient then is

$$C_D = C_{Dp} + C_{Df} \quad (90)$$



The pressure drop can be written

$$\Delta p = \Delta p^0 + \Delta p^+ \quad (91)$$

where Δp^0 is the pressure drop in the absence of the cylinder and Δp^+ is the additional pressure drop caused by the presence of the cylinder. The pressure drops are given by

$$\Delta p = p_{x=-X} - p_{x=X} \quad (92)$$

$$\Delta p^0 = p_{x=-X}^0 - p_{x=X}^0 \quad (93)$$

where the dimensionless X is a large positive number. For two-dimensional Poiseuille flow Δp^0 is

$$\Delta p^0 = \frac{12}{Re} 2X \quad (94)$$

The additional pressure drop then is

$$\Delta p^+ = \lim_{X \rightarrow \infty} (p_{x=-X} - p_{x=X} - \frac{24}{Re} X) \quad (95)$$

The force on the walls per unit length can be written as

$$F_w = F_w^0 + F_w^+ \quad (96)$$

where F_w^0 is the force on the walls in the absence of the cylinder and F_w^+ is the additional force caused by the presence of the cylinder. This force can be expressed in dimensionless form according to

$$f = F_w / (L \frac{\rho U^2}{2}) \quad (97)$$

Eq (96) then is

$$f = f^0 + f^+ \quad (98)$$

The force on the walls is calculated from eq (A54)

$$f = \frac{2}{Re} \int_{-X}^X (\omega_{y=1} - \omega_{y=-1}) dx \quad (99)$$



Because of the symmetry condition

$$\omega_{y=-1} = -\omega_{y=1} \quad (100)$$

and

$$f = \frac{4}{Re} \int_{-X}^X \omega_{y=1} dx \quad (101)$$

and

$$f^0 = \frac{4}{Re} \int_{-X}^X \omega_{y=1}^0 dx \quad (102)$$

The additional force on the walls is thus

$$f^+ = \frac{4}{Re} \int_{-\infty}^{\infty} (\omega_{y=1} - \omega_{y=1}^0) dx \quad (103)$$

For two-dimensional Poiseuille flow $\omega_{y=1}^0 = 3$ and thus

$$f^+ = \frac{4}{Re} \int_{-\infty}^{\infty} (\omega_{y=1} - 3) dx \quad (104)$$

The numerical calculations of the drag coefficient, the additional pressure drop and the additional force on the walls are quite straightforward. The equations in numerical form are consequently not written out, in order to not make the report too tedious.



7 NUMERICAL CALCULATIONS

Calculations have been performed with $d/L = 0.1$ for $Re = 0, 1, 10, 50$ and 100 . The parameters use are defined in fig. 5.

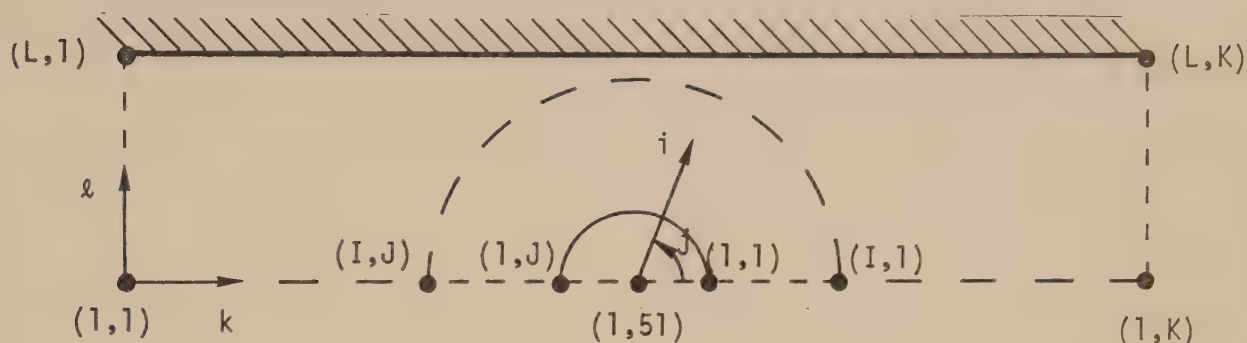


FIGURE 5. Parameters of calculations.

The calculated flow field behind the cylinder was increased with the Reynolds number, see table 1, mostly because the vorticity, generated by the cylinder, is carried by the flow far distance downstream. The mesh size was however constant throughout the calculations $h_x = h_y = 0.04$ and thus $L = 26$. The "inner" field was always the same $I = 51, J = 31, h_\xi = 0.014$ and $h_\eta = 1/30$.

Re	K	ρ_ω
0	151	1.8
1	151	1.8
10	251	1.2
50	351	0.75
100	351	0.50

TABLE 1. Parameters of calculations.

When the Reynolds number is increased the "non-linearity" of the vorticity equations is increased. Hence the relaxation factor has to be decreased, in order to not get the equations unstable, see table 1. In the equations for the streamfunction and the pressure the relaxation factor was independent on Reynolds number, $\rho_\psi = \rho_p = 1.8$.



The iterations of the equations started with rather rough convergence criterions. These were refined as the iterations proceeded and terminated with $\varepsilon_{\omega} = \varepsilon_{\psi} = \varepsilon_p = 0.001$.

The calculations have been performed on the computer UNIVAC 1108 at Lund Computing Centre, Lund.



8 RESULTS

The deviation of the flow from the Stokes' régime ($Re = 0$), in which upstream/downstream symmetry prevails, is best illustrated by the isovorticity lines and isobars. Streamlines and isovorticity lines are shown in figs 6 - 10 and isobars in figs 11 - 15 for Reynolds numbers 0, 1, 10, 50 and 100. The figures show clearly how the generated vorticity is carried considerable distances downstream at the higher Reynolds numbers. For $Re = 50$ and 100 there is also a remarkable difference in the streamline pattern from the lower Reynolds numbers. As Re is increased a stage is reached when a standing circulating eddy appears at the back of the cylinder, the flow separates. The back-flow begins at the point where $\omega = 0$ on the surface, i.e. the point of zero wall friction. The Reynolds number for which separation first occurs is the critical Reynolds number of separation and it is determined of the criterion $\frac{\partial \omega}{\partial \theta} = 0$ on the surface for $\theta = 0$. The critical Reynolds number in this case is estimated to 45.5 (interpolated), corresponding to the critical particle Reynolds number 6.8. The particle Reynolds number is defined by

$$Re_p = \frac{\frac{3}{2} \cdot U \cdot d \cdot \rho}{\mu} = \frac{3}{2} \frac{d}{L} Re \quad (105)$$

For an unbounded flow around a cylinder the calculated critical Reynolds number scatters about $Re = 40$ ($Re_p = 6$). The wake length (measured from the rear end of the cylinder) for $Re = 50$ is 0.003 and for $Re = 100$ it is 0.111. The angle of separation is 14.0° and 38.2° for $Re = 50$ and 100 respectively. Close-ups of the streamline, isovorticity and isobar patterns for $Re = 100$ are shown in figs 16 and 17.

The vorticity distribution on the surface of the cylinder (which is closely related to the local friction stress, eq (88)) is shown in fig 18.

Figure 19 shows the calculated pressure distribution on the cylinder surface. For low Reynolds numbers the minimum pressure occurs at the rear stagnation point, whereas it occurs on the sides of

the cylinder for higher Reynolds numbers, as would be expected from Bernouille's theorem if the fluid was inviscid.

The calculated drag coefficients, C_D , C_{Df} , C_{Dp} , the pressure difference over the cylinder, $p_\pi - p_o$, and the calculated additional pressure drop, Δp^+ , and the wall force, f^+ , caused by the presence of the cylinder are summarized in table 2 for various values of Reynolds number.

Re	C_{Df}	C_{Dp}	C_D	$p_\pi - p_o$	Δp^+	f^+
0	129.4/Re	137.6/Re	267.0/Re	176.1/Re	39.93/Re	13.16/Re
1	129.5	137.8	267.3	176.3	39.98	13.20
10	13.5	14.6	28.1	18.8	4.23	1.41
50	3.67	4.57	8.24	6.25	1.25	0.427
100	2.28	3.24	5.52	4.75	0.837	0.285

TABLE 2. Calculated properties.

A check upon the reliability of calculated datas is obtained by the momentum balance. The "additional momentum balance", eq (A62), is in dimensionless form

$$\Delta p^+ - f^+ = r_o C_D \quad (106)$$

The calculated values satisfy this condition very well.

The agreement with Faxen's [63] values of C_D and Δp^+ for $Re = 0$ is also very good ($C_D = 267.2/Re$, $\Delta p^+ = 39.88/Re$). The "momentum balance", eq (106), applied on Faxen's data gives $f^+ = 13.16/Re$.

The influence of the walls on the drag on the cylinder can be estimated by a wall correction factor

$$\alpha_D = \frac{C_D}{C_{Dunbounded}} \quad (107)$$



In fig 20 is the calculated α_D shown as a function of Re . Values of $C_{Dunbounded}$ are taken from a curve, which is fitted to a large number of numerically calculated drag coefficients. No value of α_D can be obtained for $Re_p = 0$ because there is no solution to the unbounded flow around a cylinder in creeping motion (Stokes' paradox). For low Reynolds numbers Lamb's solution [4] gives

$$\alpha_D = 2.76 - 0.708 \ln Re \quad (108)$$

From fig 20 it is seen that the wall effects become negligible for $Re > 100$.

Brenner [67], [68] has derived a relationship between the additional pressure drop caused by a sphere and the drag on a sphere in a closed conduit. A similar approach (see Appendix) leads to the equation

$$\frac{\Delta p^+}{r_o C_D} = 1.5 \quad (109)$$

for Poiseuille flow around a cylinder halfway between two parallel plates, when d/L is small. This relationship is valid not only for low Reynolds numbers, but for the whole Re -range investigated.



STREAMLINES AND ISOVORTICITY LINES FOR $Re = 0$

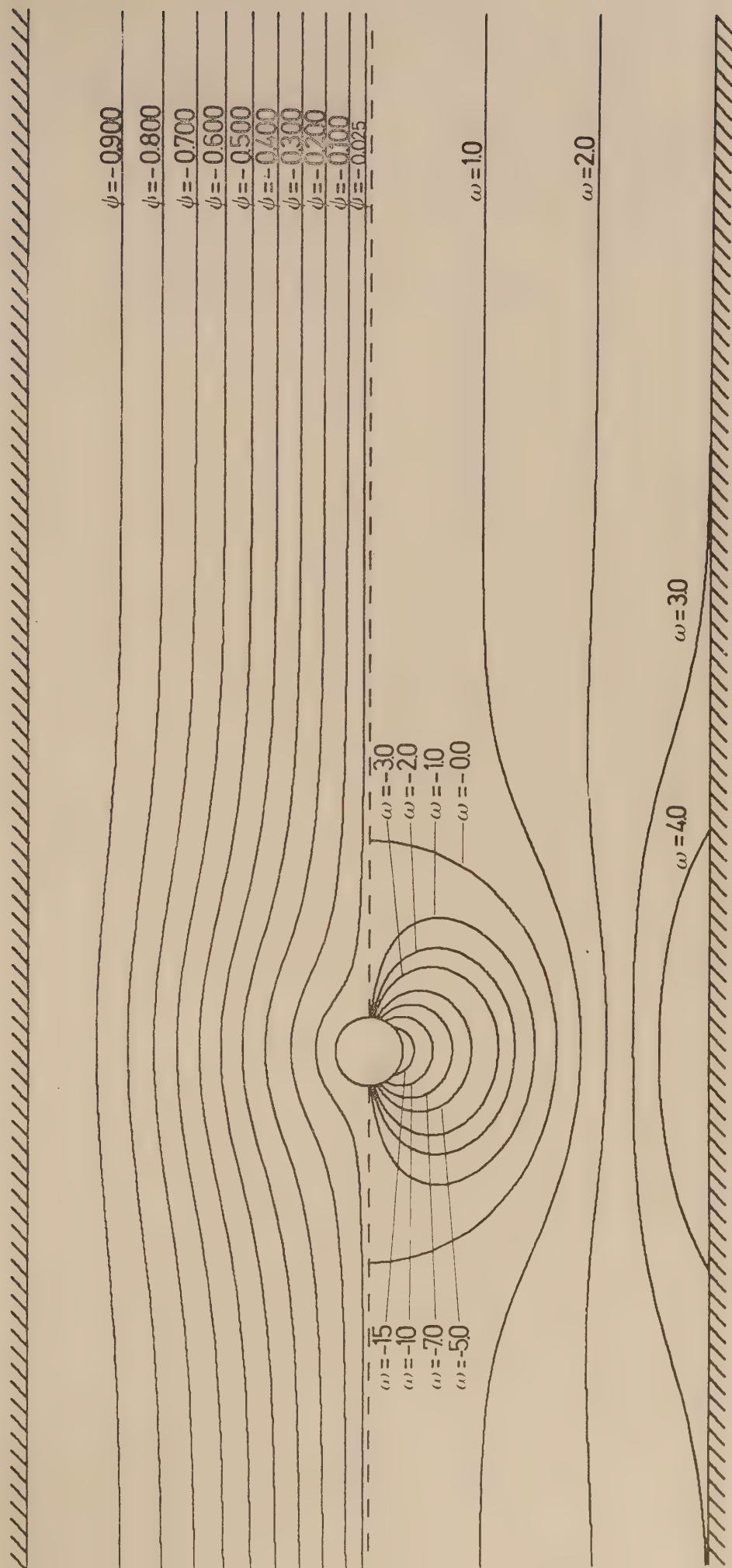
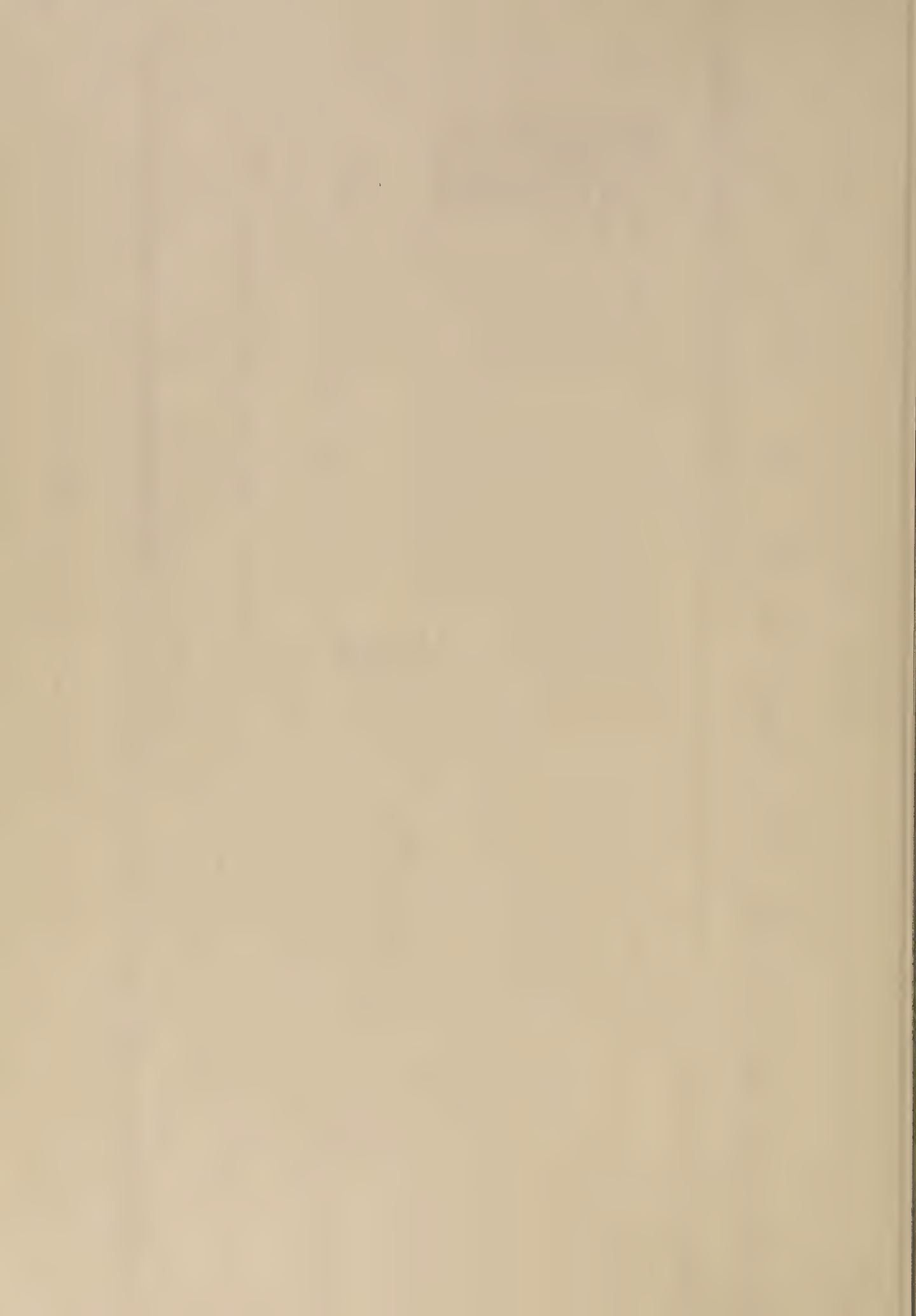


FIGURE 6. Streamlines and isovorticity lines for $Re = 0$.



STREAMLINES AND ISOVORTICITY LINES FOR $Re = 1$

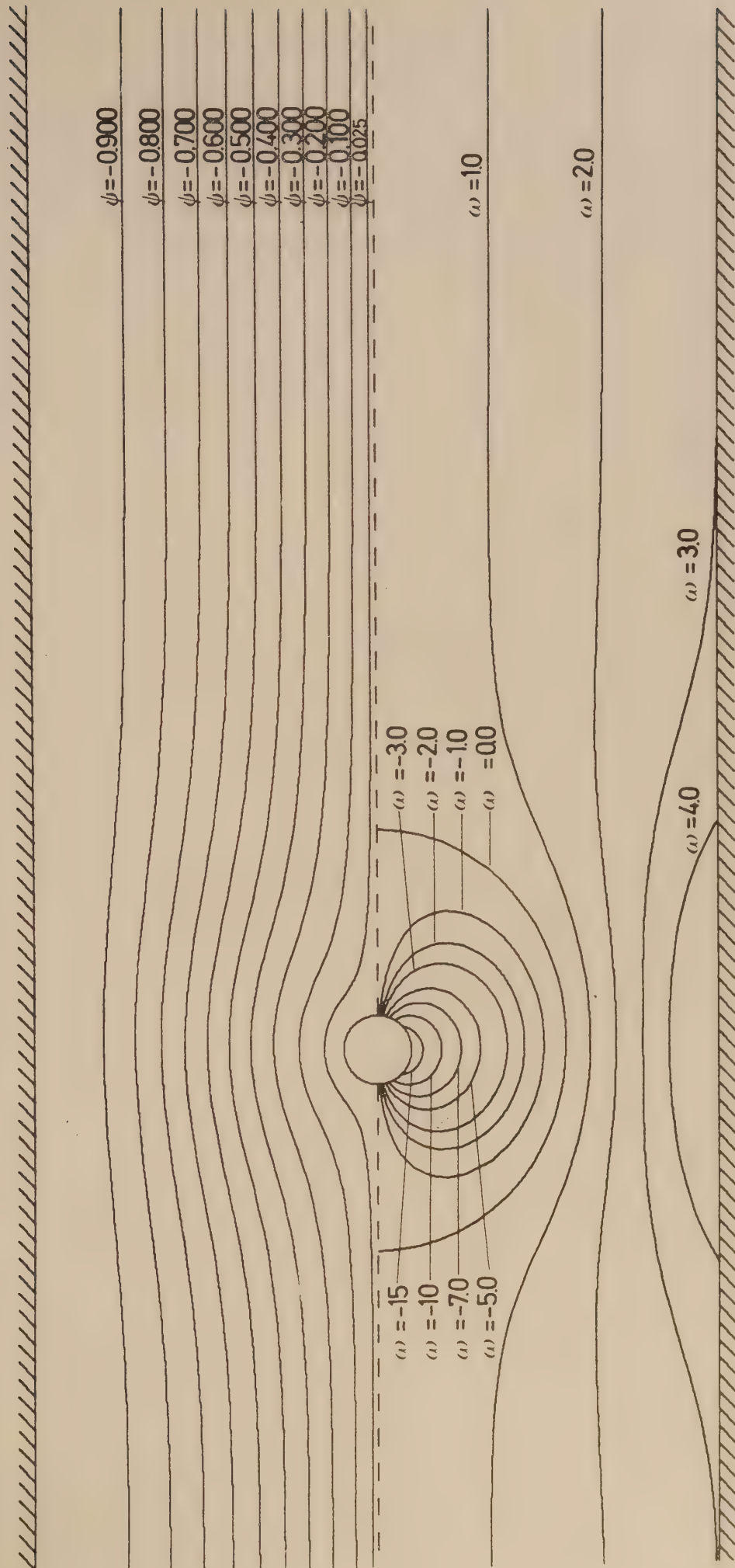


FIGURE 7. Streamlines and isovorticity lines for $Re = 1$.



STREAMLINES AND ISOVORTICITY LINES FOR $Re = 10$

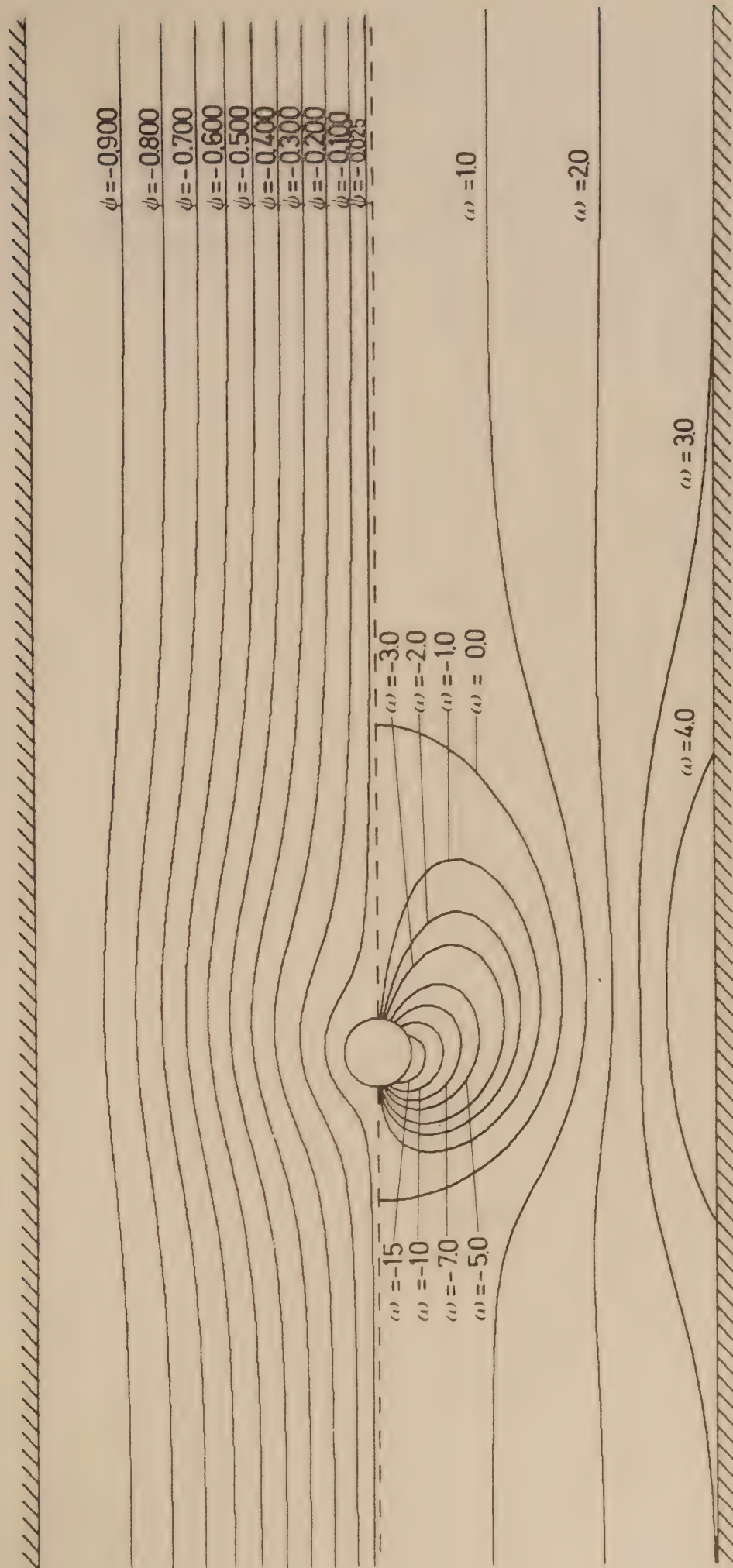


FIGURE 8. Streamlines and isovorticity lines for $Re = 10$.



STREAMLINES AND ISOVORTICITY LINES FOR $Re = 50$

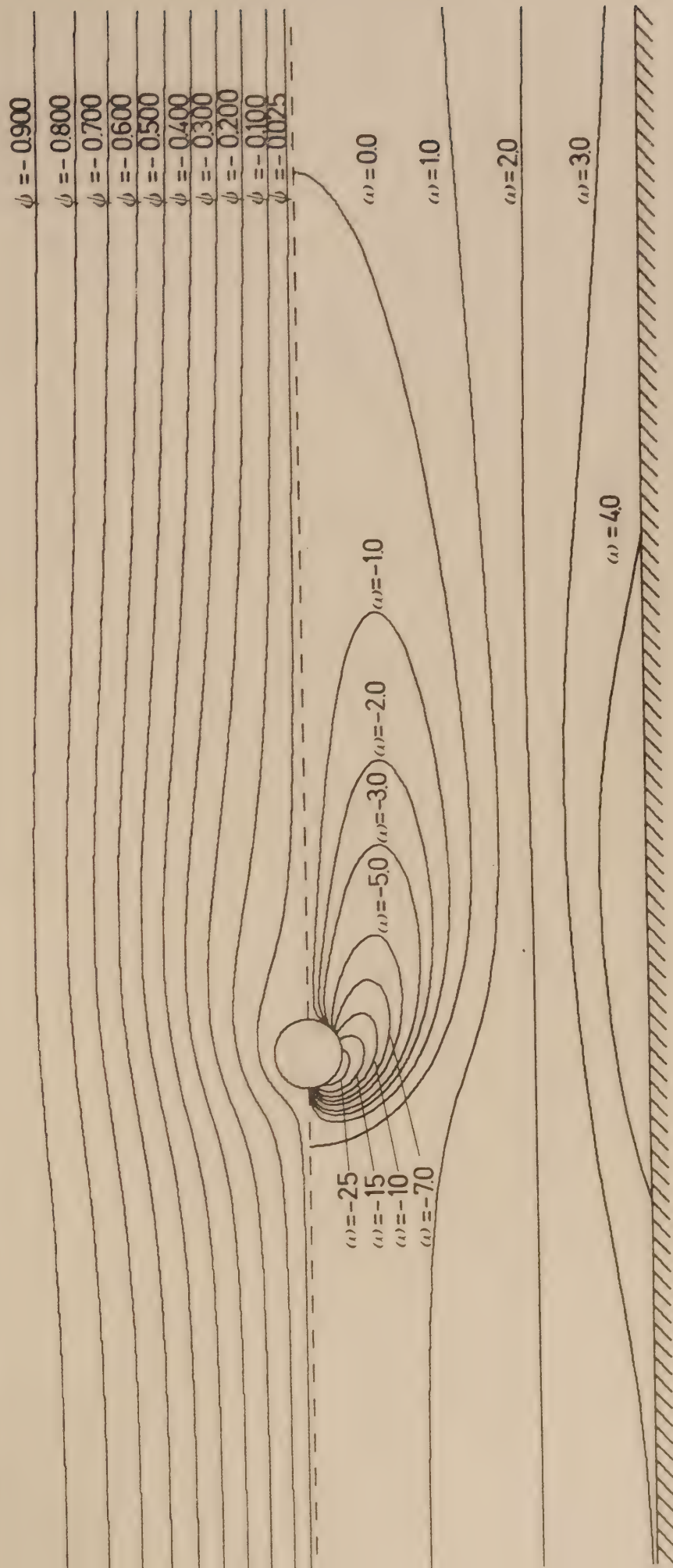


FIGURE 9. Streamlines and isovorticity lines for $Re = 50$.



STREAMLINES AND ISOVORTICITY LINES FOR $Re = 100$

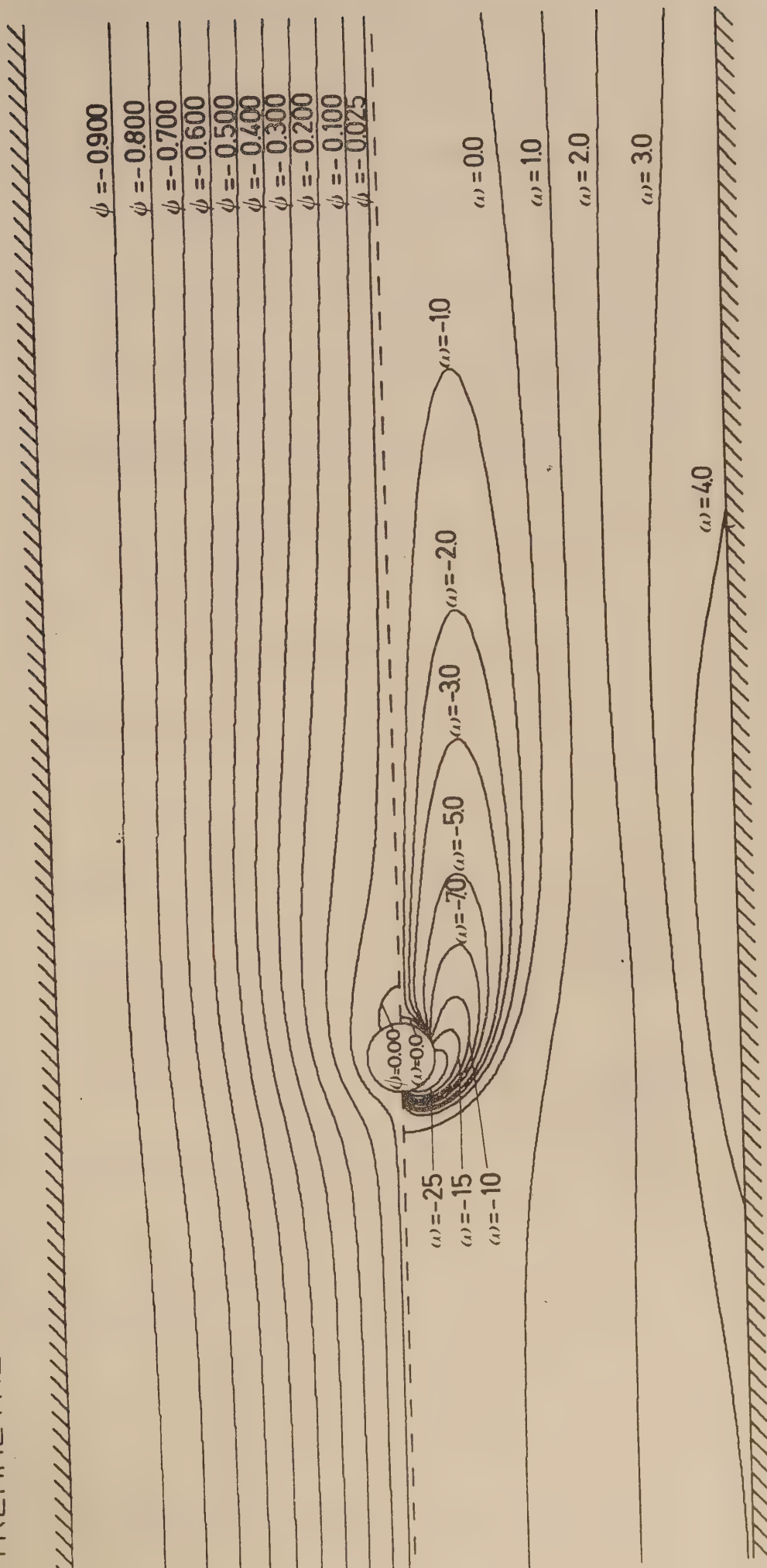
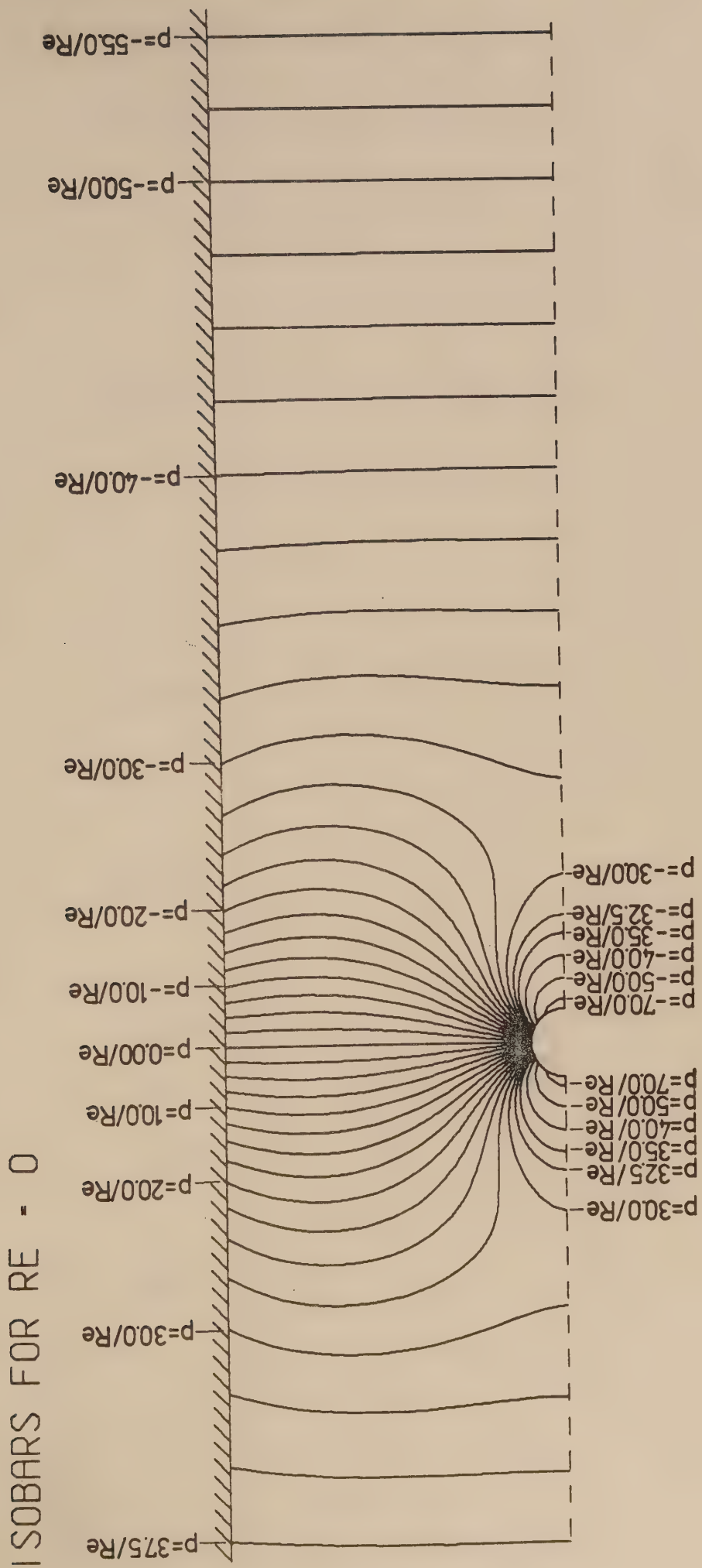
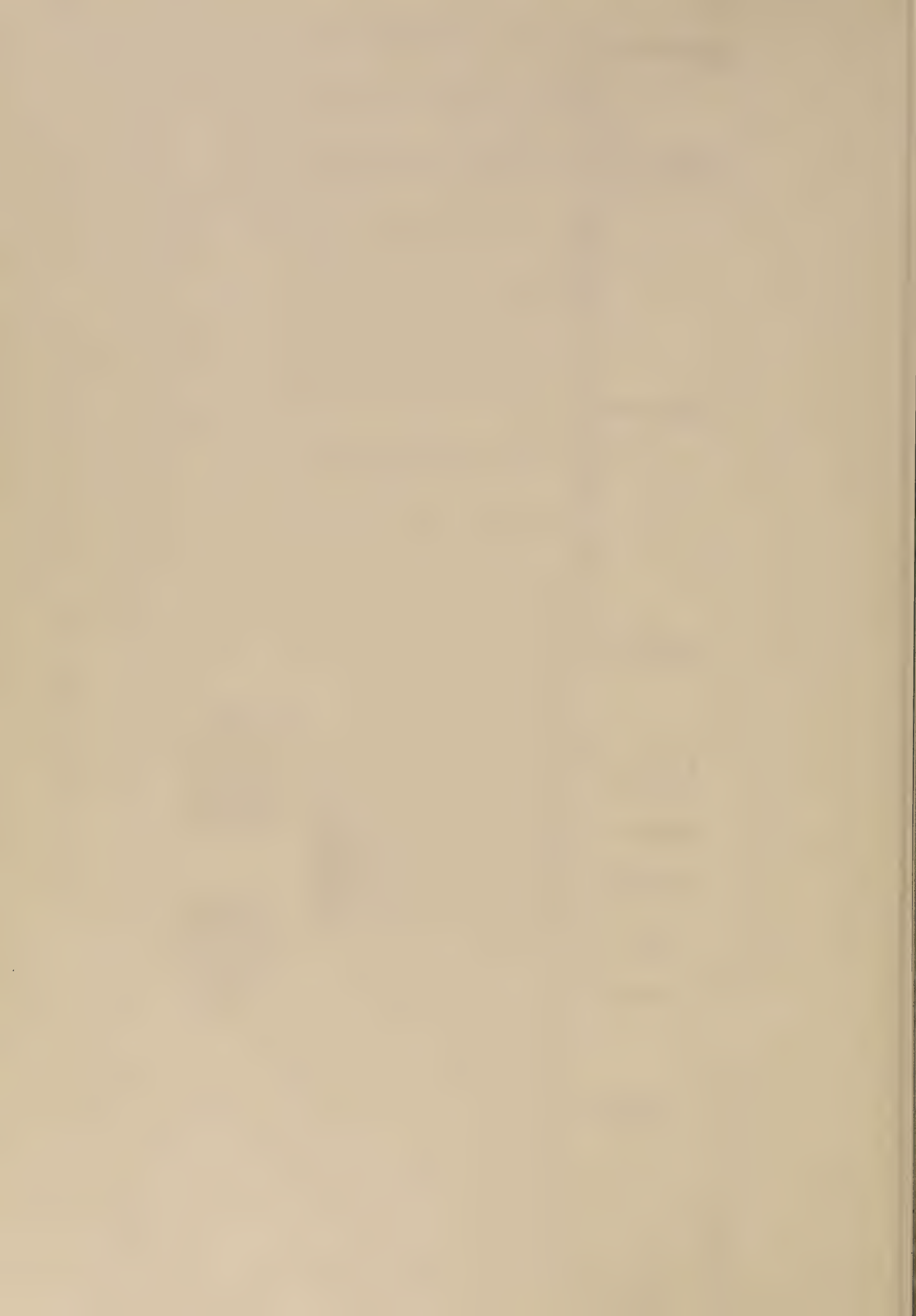


FIGURE 10. Streamlines and isovorticity lines for $Re = 100$.



FIGURE 11. Isobars for $Re = 0$.



ISOBARS FOR $Re = 1$

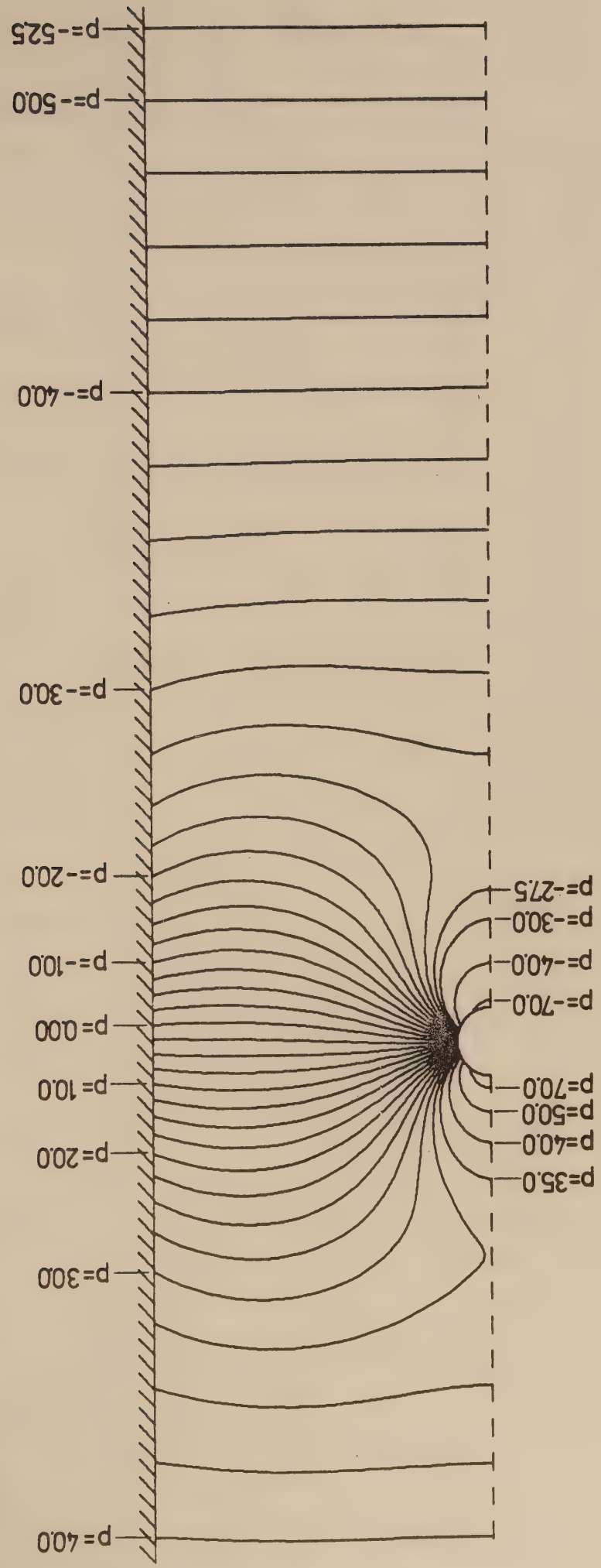


FIGURE 12. Isobars for $Re = 1$.



SOBARS FOR RE = 10

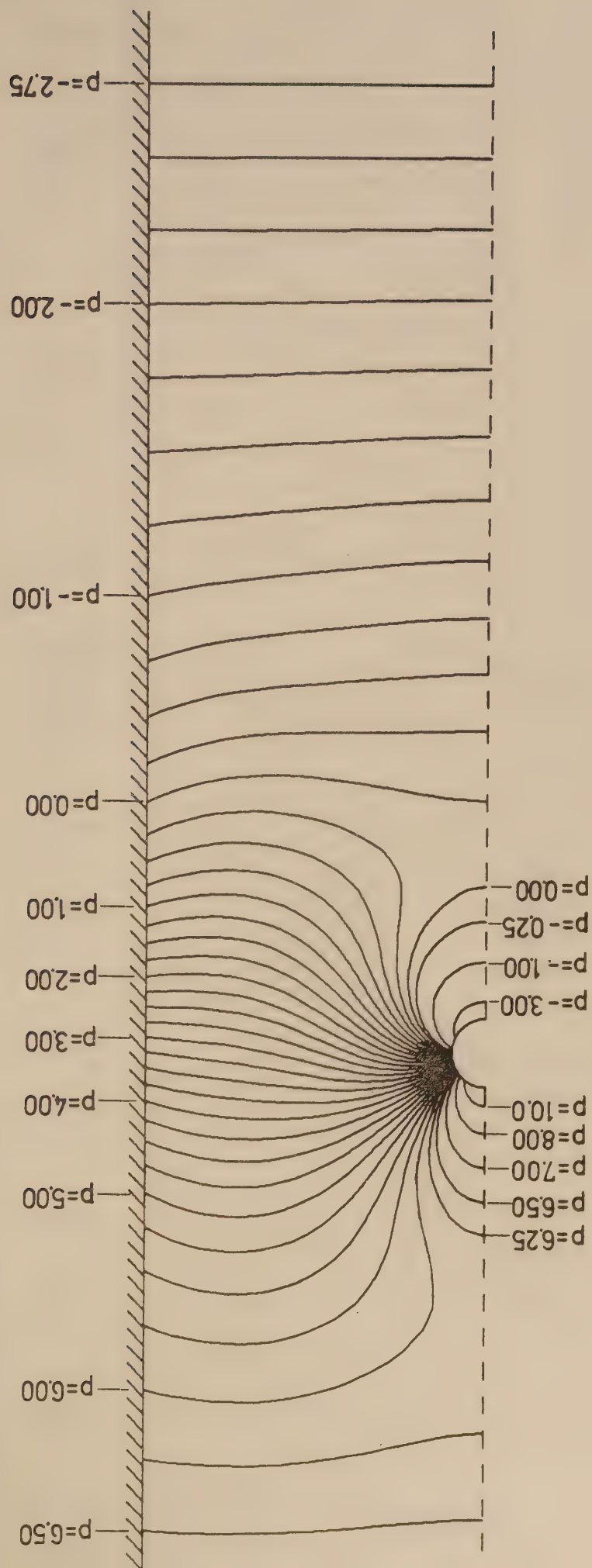
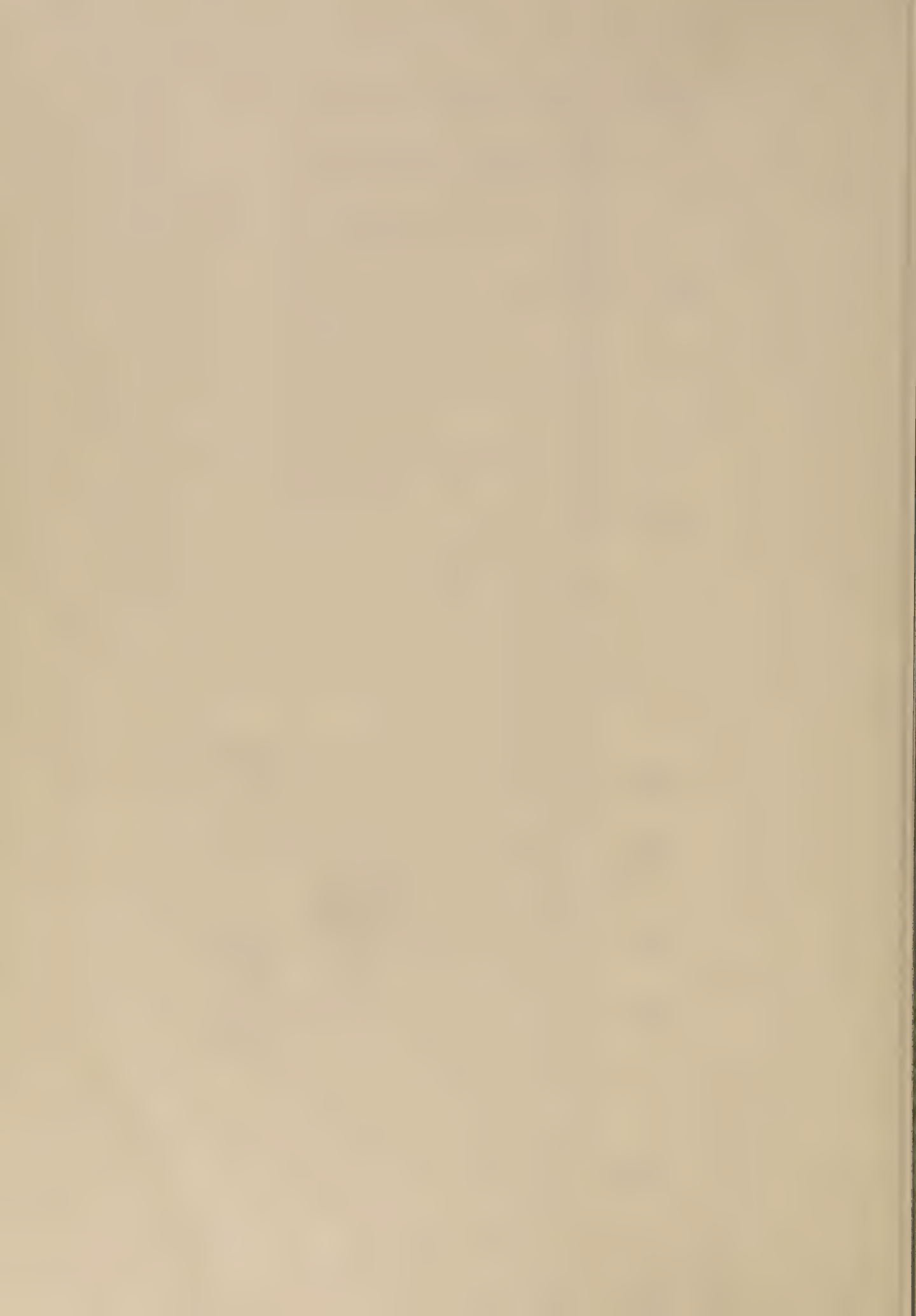


FIGURE 13. Isobars for $Re = 10$.



ISOBARS FOR $Re = 50$

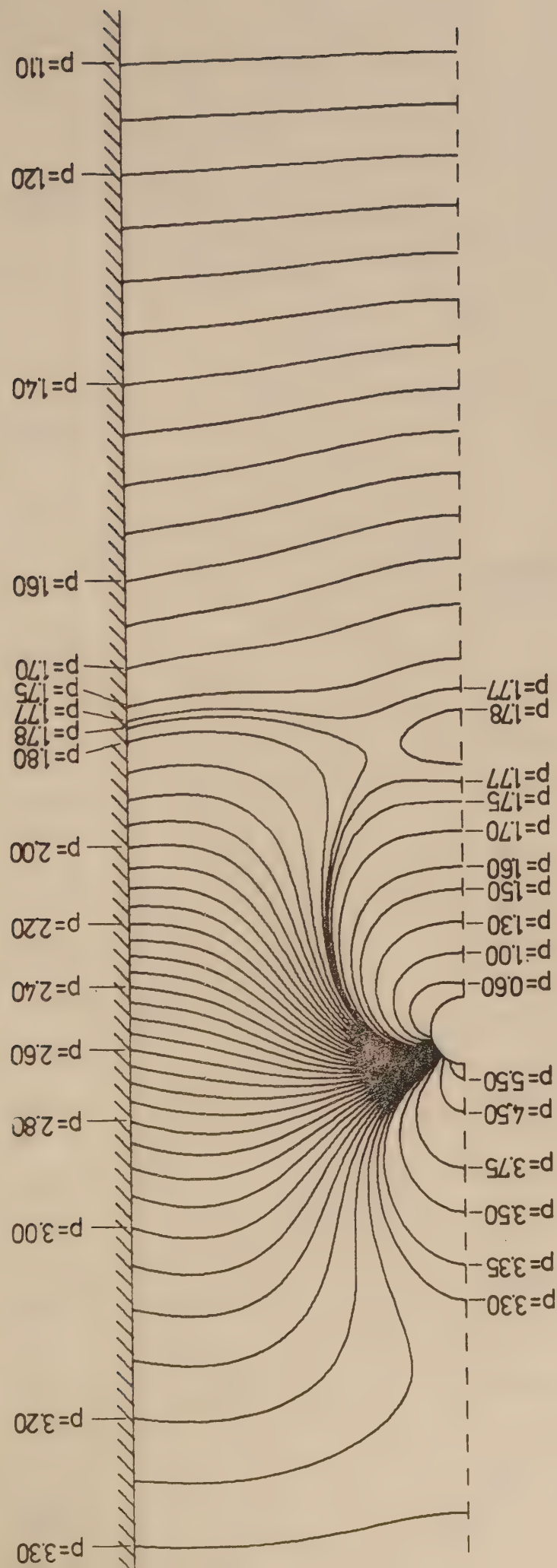
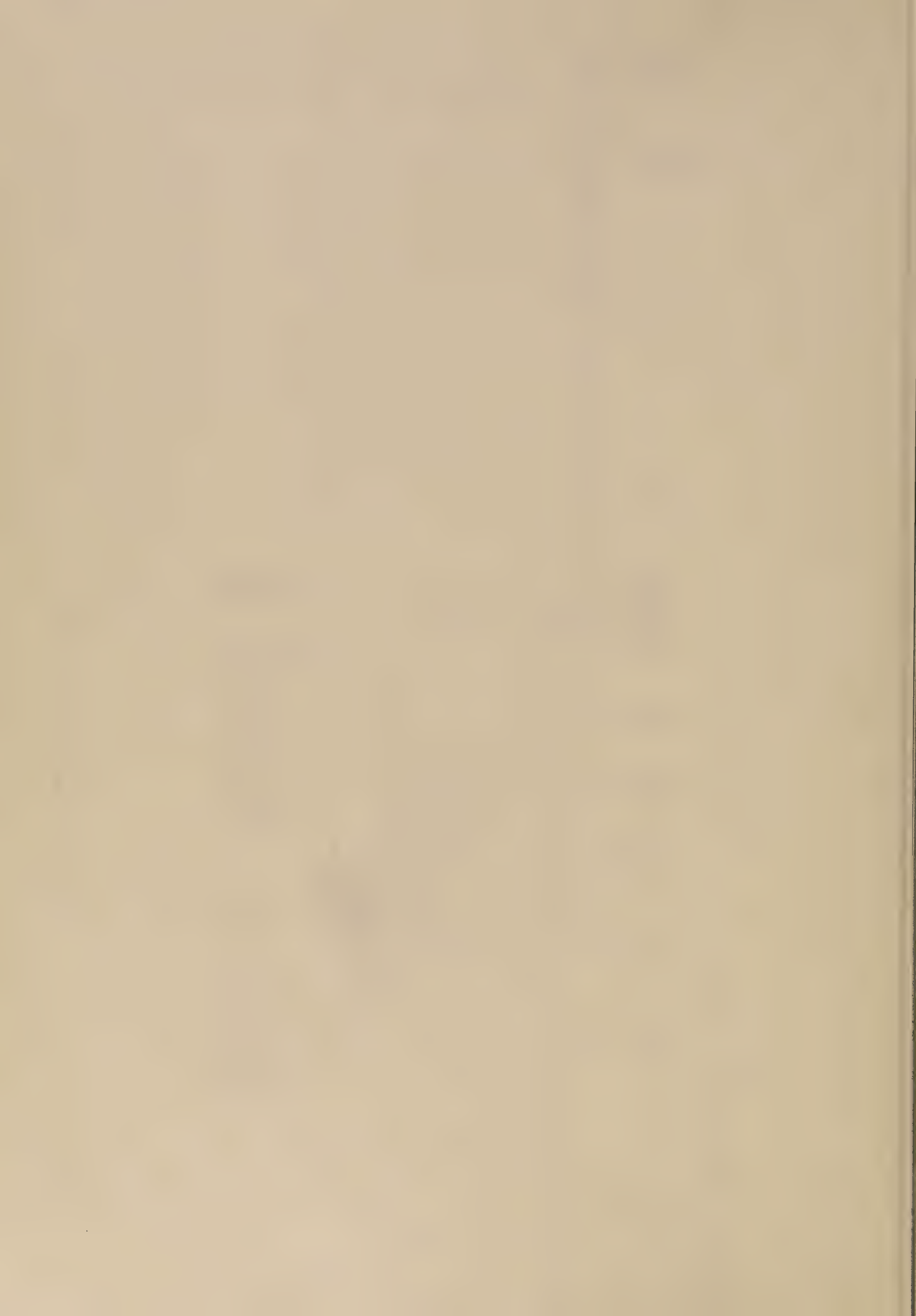


FIGURE 14. Isobars for $Re = 50$.



ISOBARS FOR $Re = 100$

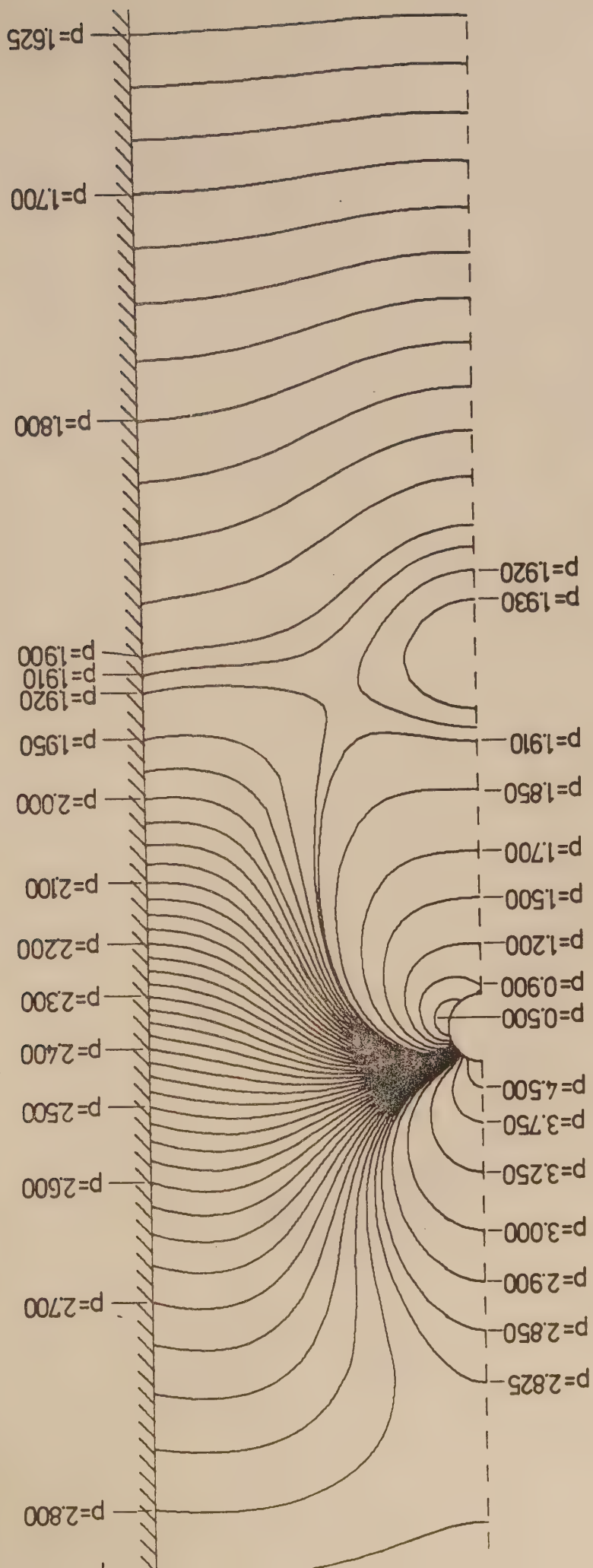
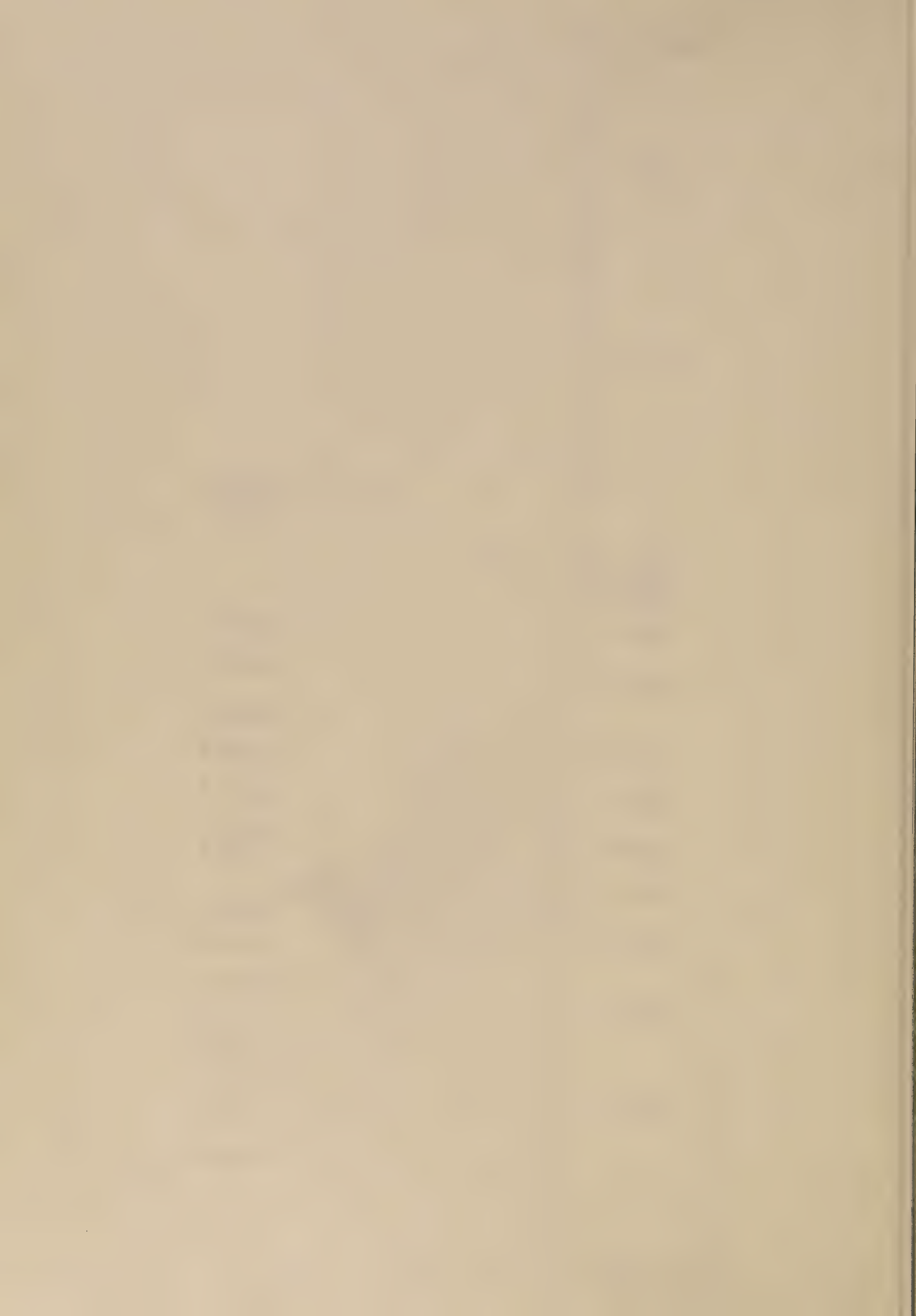


FIGURE 15. Isobars for $Re = 100$.



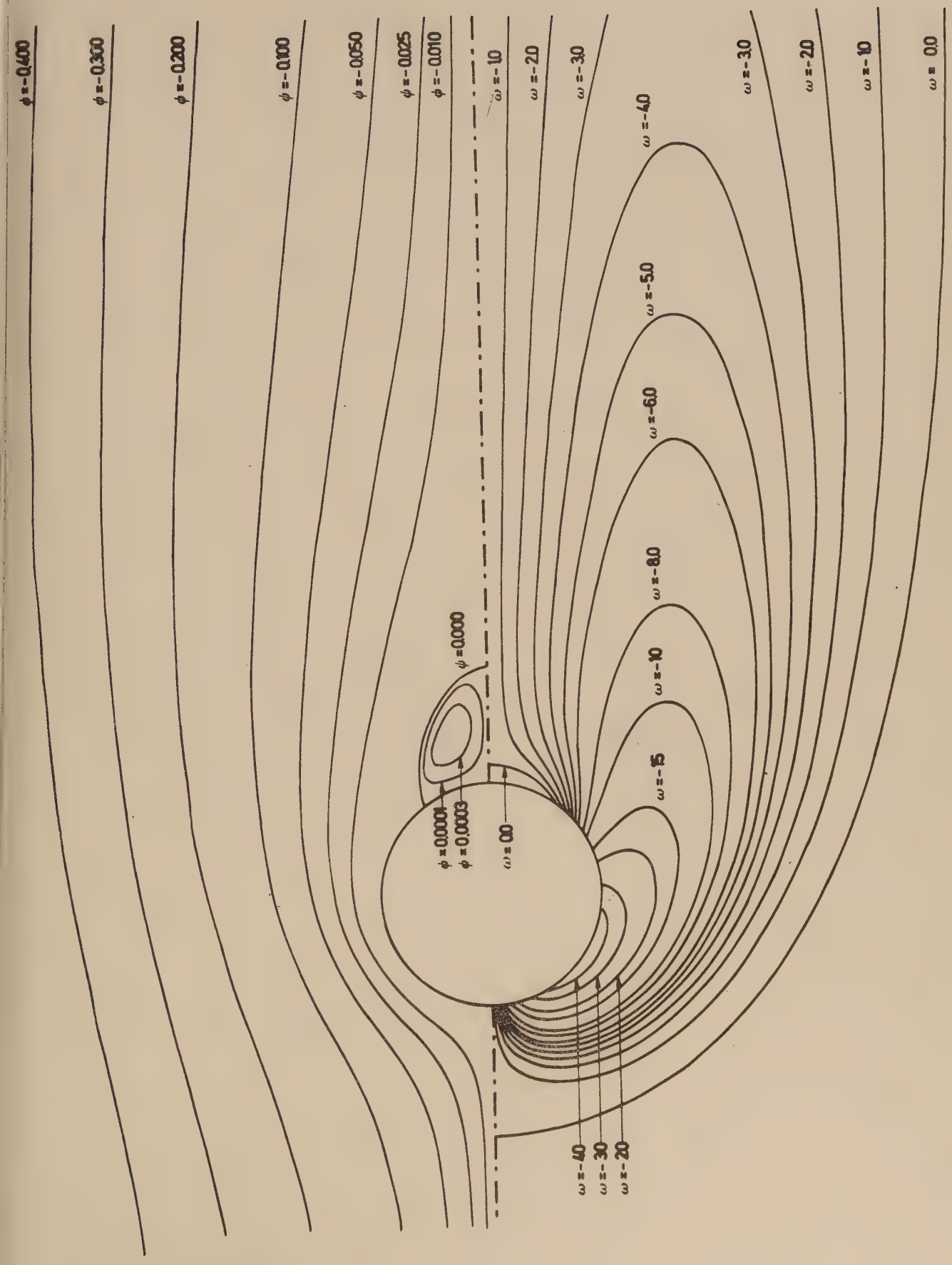
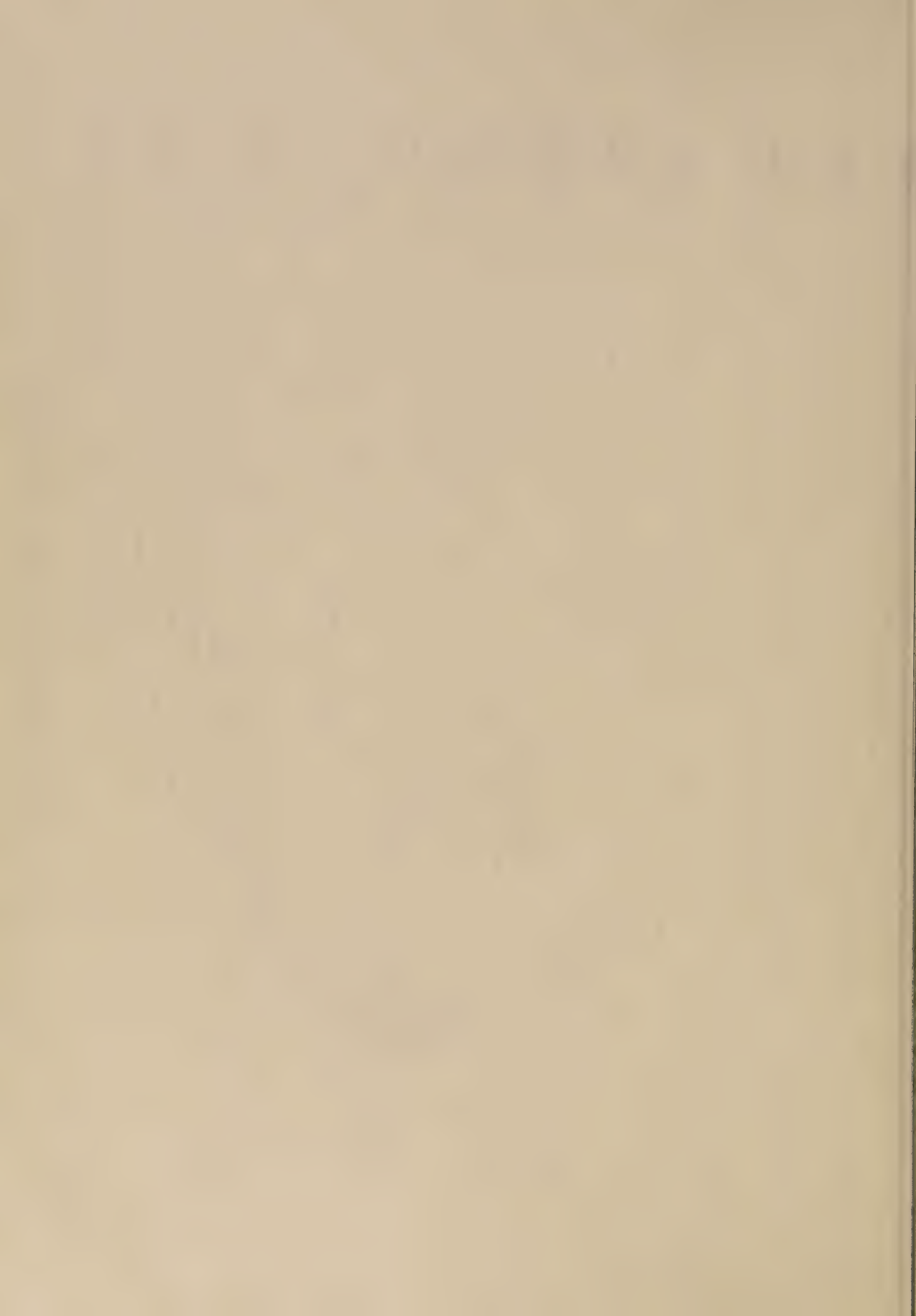


FIGURE 16. Streamlines and isovorticity lines close to the cylinder for $Re = 100$.



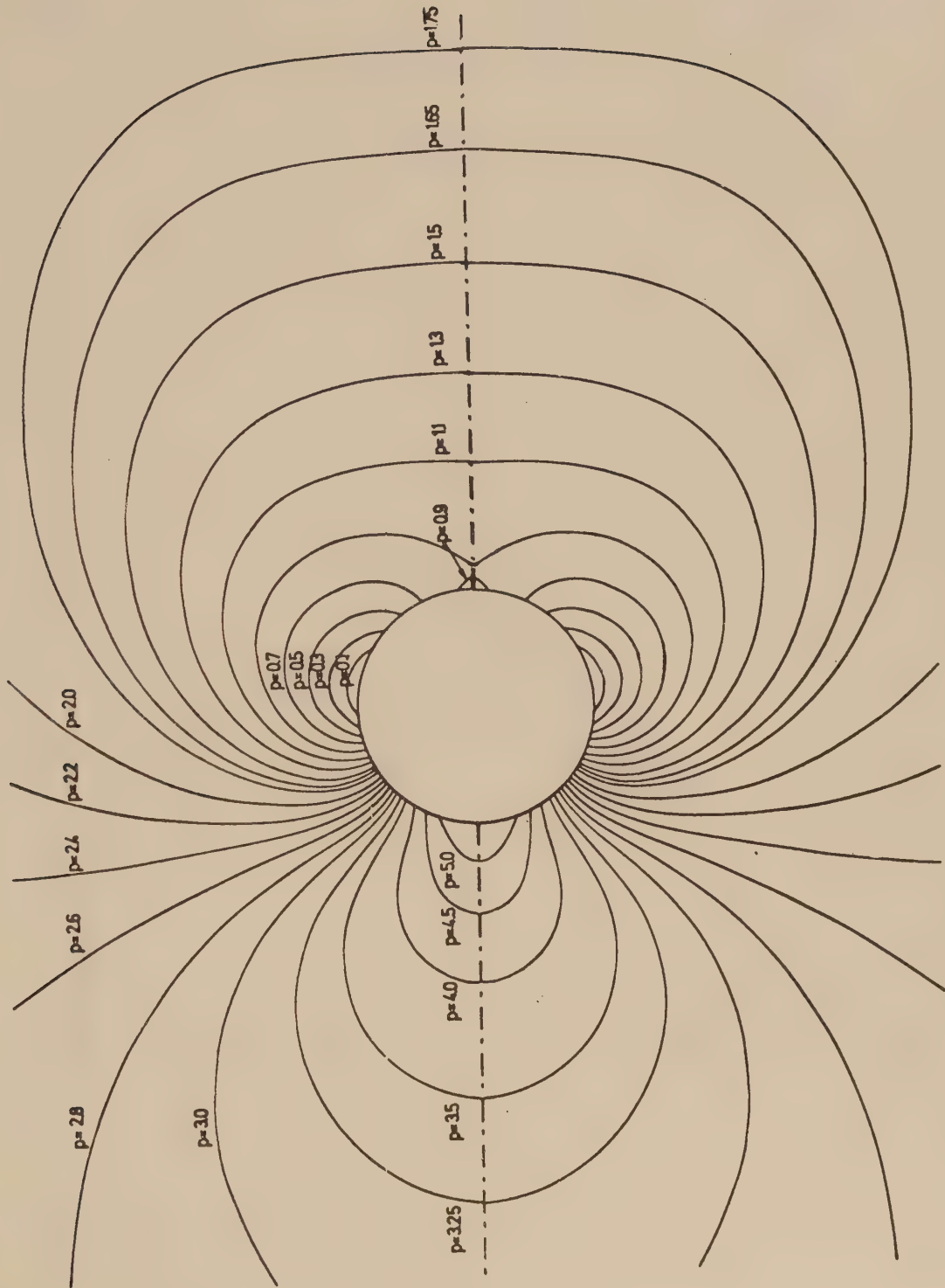
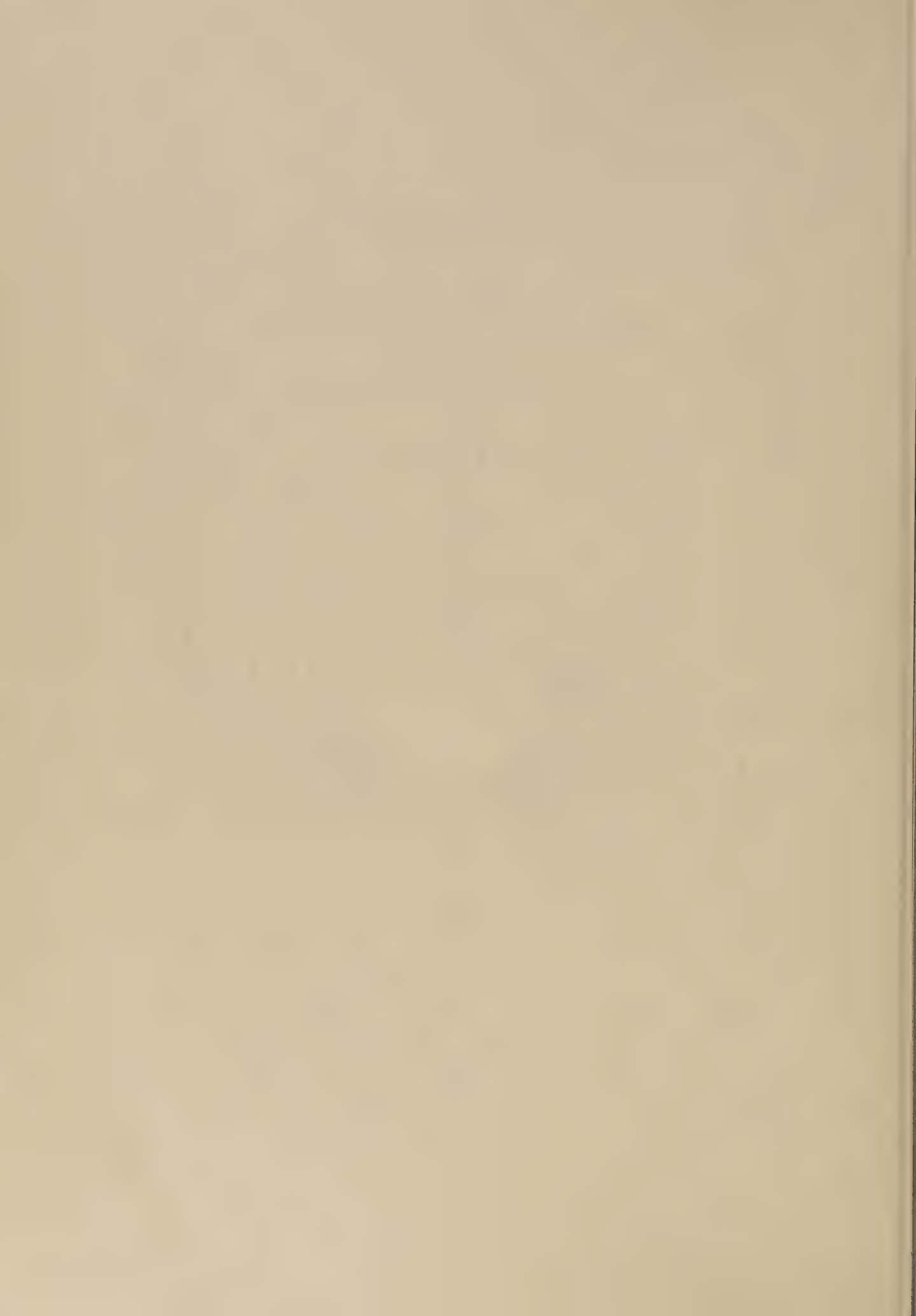


FIGURE 17. Isobars close to the cylinder for $Re = 100$.



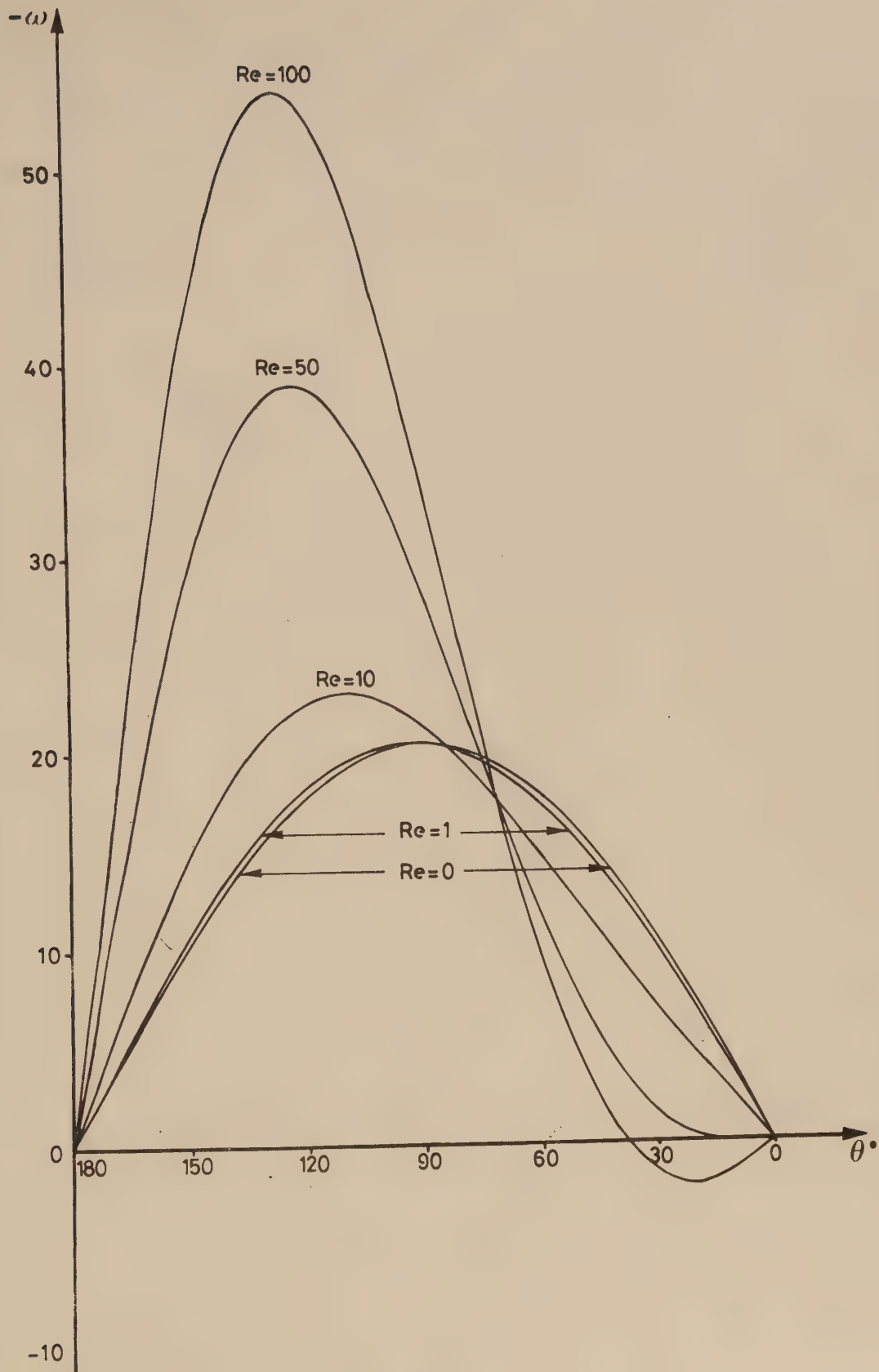


FIGURE 18. The vorticity distribution on the surface of the cylinder.



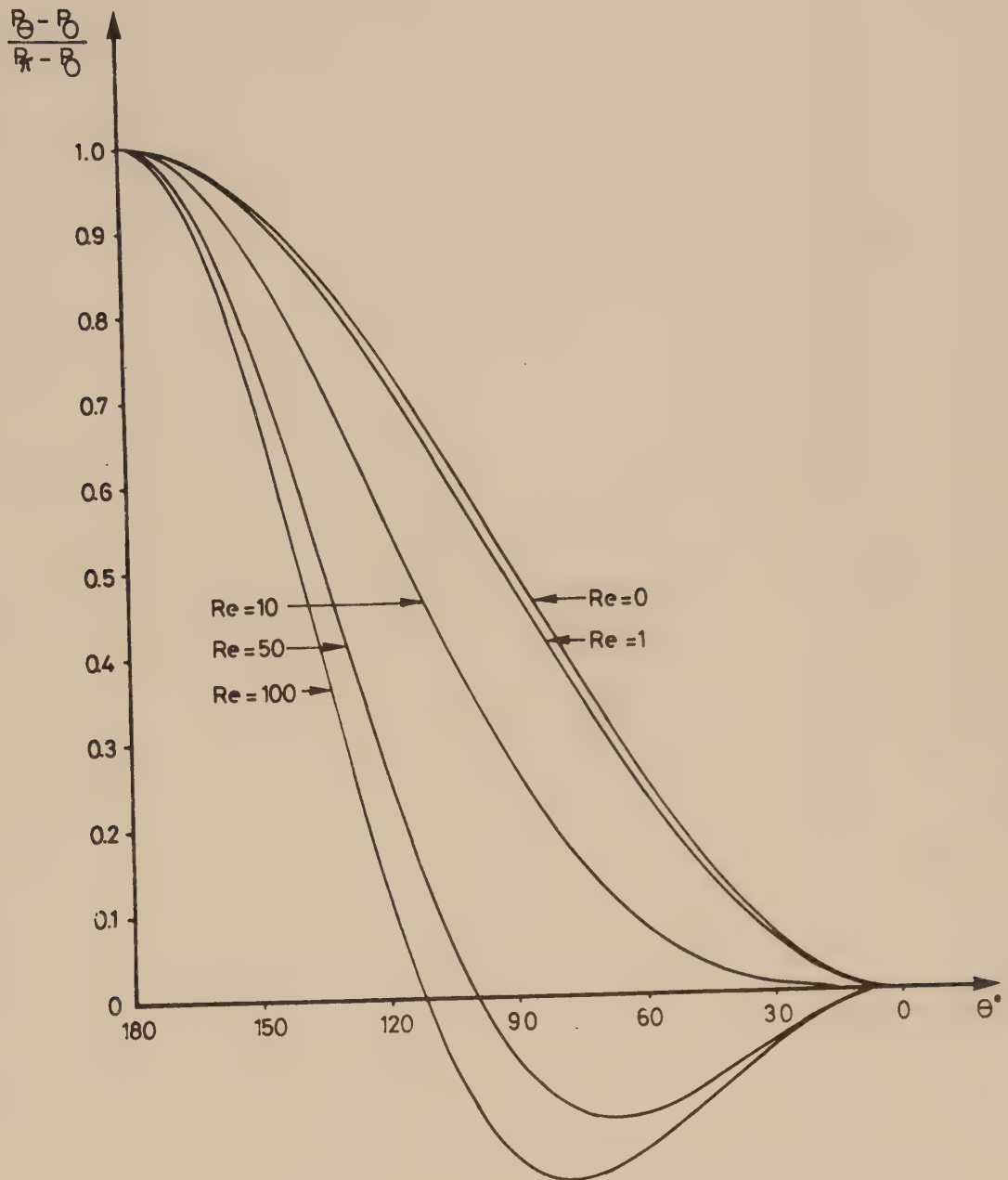


FIGURE 19. The pressure distribution on the surface of the cylinder.

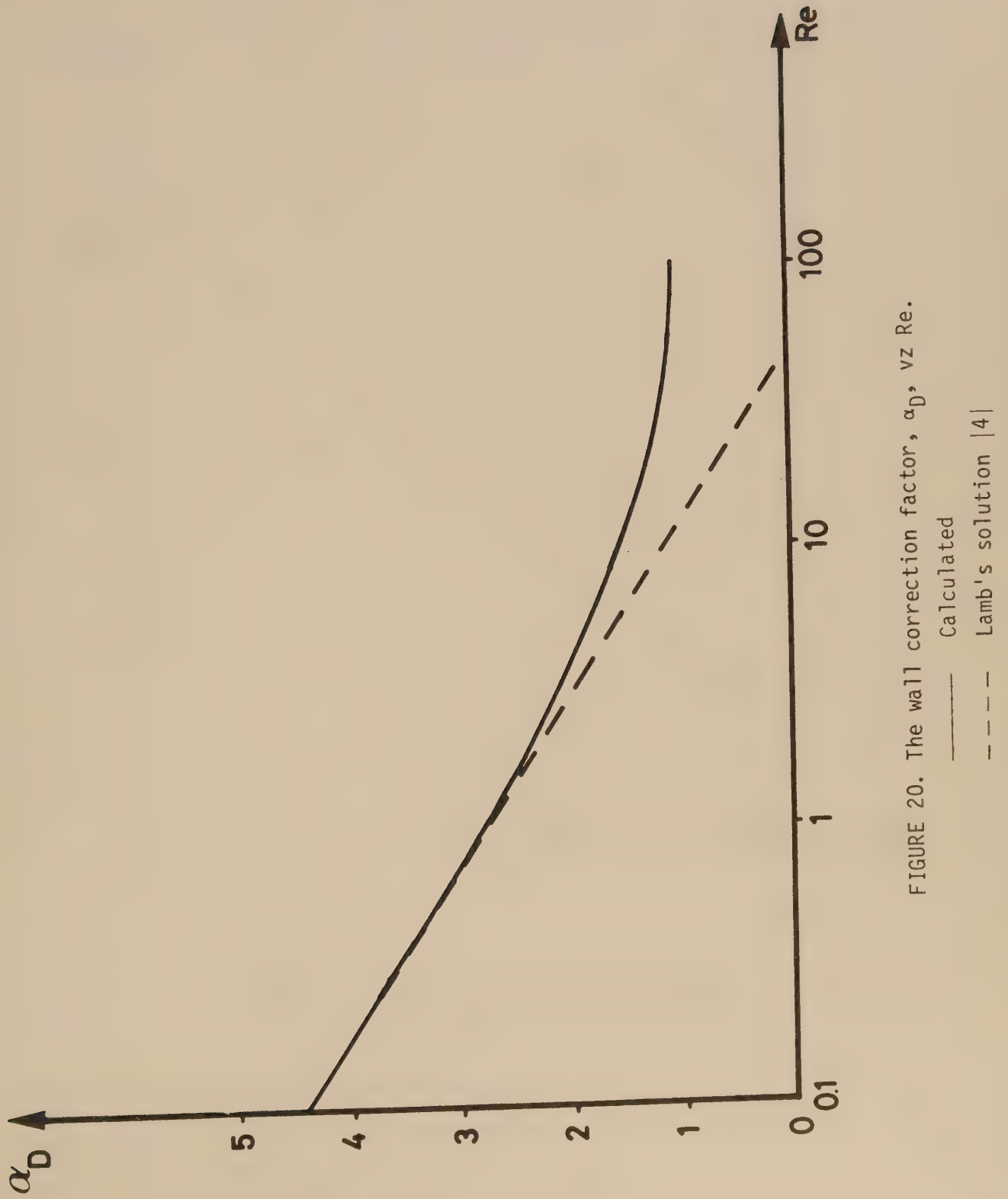
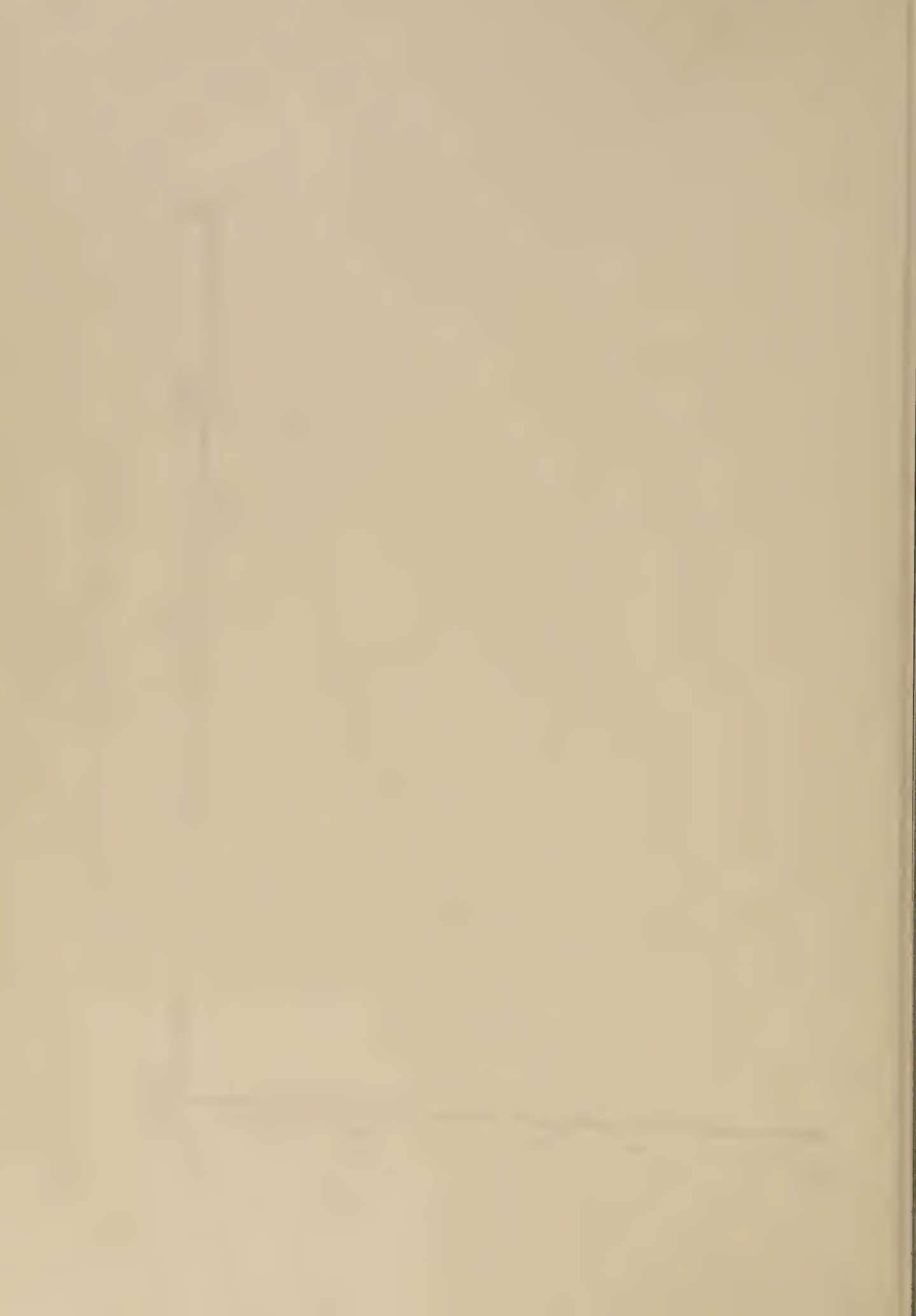


FIGURE 20. The wall correction factor, α_D , vs Re .



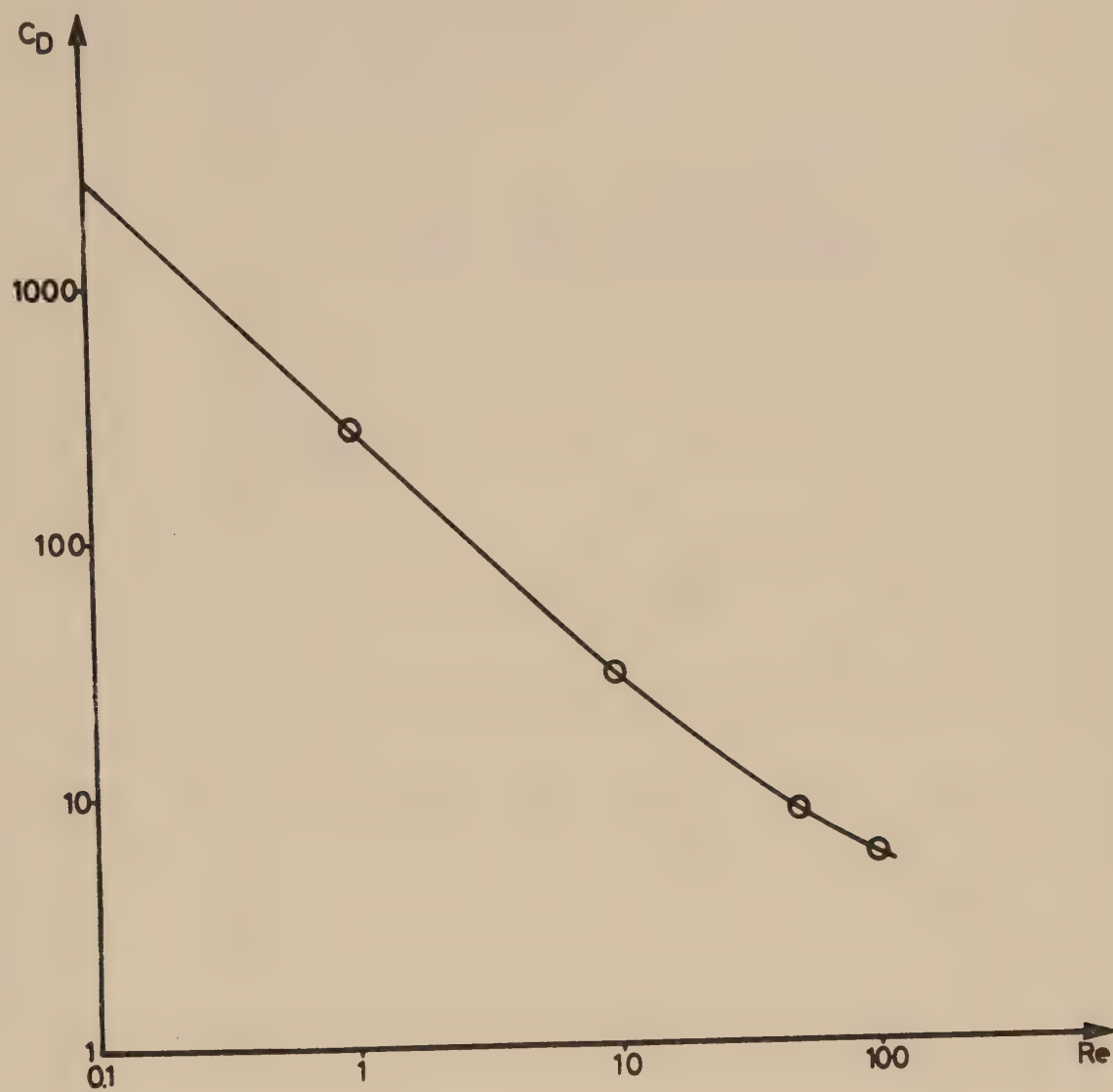


FIGURE 21. The calculated drag coefficient, C_D , vs Re .



NOTATIONS

C_D	drag coefficient, def. in eq (85)
C_{Df}	friction drag coefficient
C_{Dp}	pressure drag coefficient
d	cylinder diameter
E	def. in eq (14)
	or
	rate of mechanical energy dissipation
f	wall force, dimensionless, def. in eq (97)
F	force
F_D	drag force
F_{Df}	friction drag force
F_{Dp}	pressure drag force
F_w	wall force
h	grid spacing
i	grid-point index ("inner" field)
$\bar{i}$	unit vector
I	number of grid-points in ξ -direction ("inner" field)
$\bar{I}$	idem factor
j	grid-point index ("inner" field)
J	number of grid-points in η -direction ("inner" field)
k	grid-point index ("outer" field)
K	number of grid-points in x -direction ("outer" field)
ℓ	grid-point index ("outer" field)
L	number of grid-points in y -direction ("outer" field)
	or
	channel width
m	number of iterations
n	number of iterations
p	pressure [†] , dimensionless with respect to $\frac{1}{2} \rho U^2$
Δp	pressure drop [†] , dimensionless with respect to $\frac{1}{2} \rho U^2$
r	cylindrical coordinate [†] , dimensionless with respect to $L/2$
r_0	cylinder radius, dimensionless with respect to $L/2$
R	cylinder radius
Re	channel Reynolds number, def. in eq (5)
Re_p	particle Reynolds number, def. in eq (105)

[†] not dimensionless in Appendix



S	surface area
$d\bar{S}$	directed element of surface area
U	mean velocity in the channel
v	velocity [†] , dimensionless with respect to U
V	volume
x	cartesian coordinate [†] , dimensionless with respect to $L/2$
X	large positive number
y	cartesian coordinate [†] , dimensionless with respect to $L/2$
z	cartesian coordinate [†] , dimensionless with respect to $L/2$
α_D	wall correction factor, def. in eq (107)
δ_{ij}	unit tensor
$\bar{\Delta}$	rate of deformation tensor, def. in eq (A36)
ϵ	convergence parameter
η	cylindrical coordinate, def. in eq (10)
θ	cylindrical coordinate
μ	dynamic viscosity
ξ	cylindrical coordinate, def. in eq (9)
$\bar{\Pi}$	stress tensor, def. in eq (A35)
ρ	density
	or
	relaxation factor
Φ	def. in eqs (68) and (75)
	or
	local rate of energy dissipation per unit volume
ψ	streamfunction [†] , dimensionless with respect to $\frac{U L}{2}$
ω	vorticity [†] , dimensionless with respect to $\frac{2 U}{L}$
∇	differential operator
∇^2	Laplace operator [†] , dimensionless with respect to $\frac{4}{L^2}$

Superscripts

m	iteration index
n	iteration index
new	iteration index
old	iteration index
T	transpose
o	unperturbated

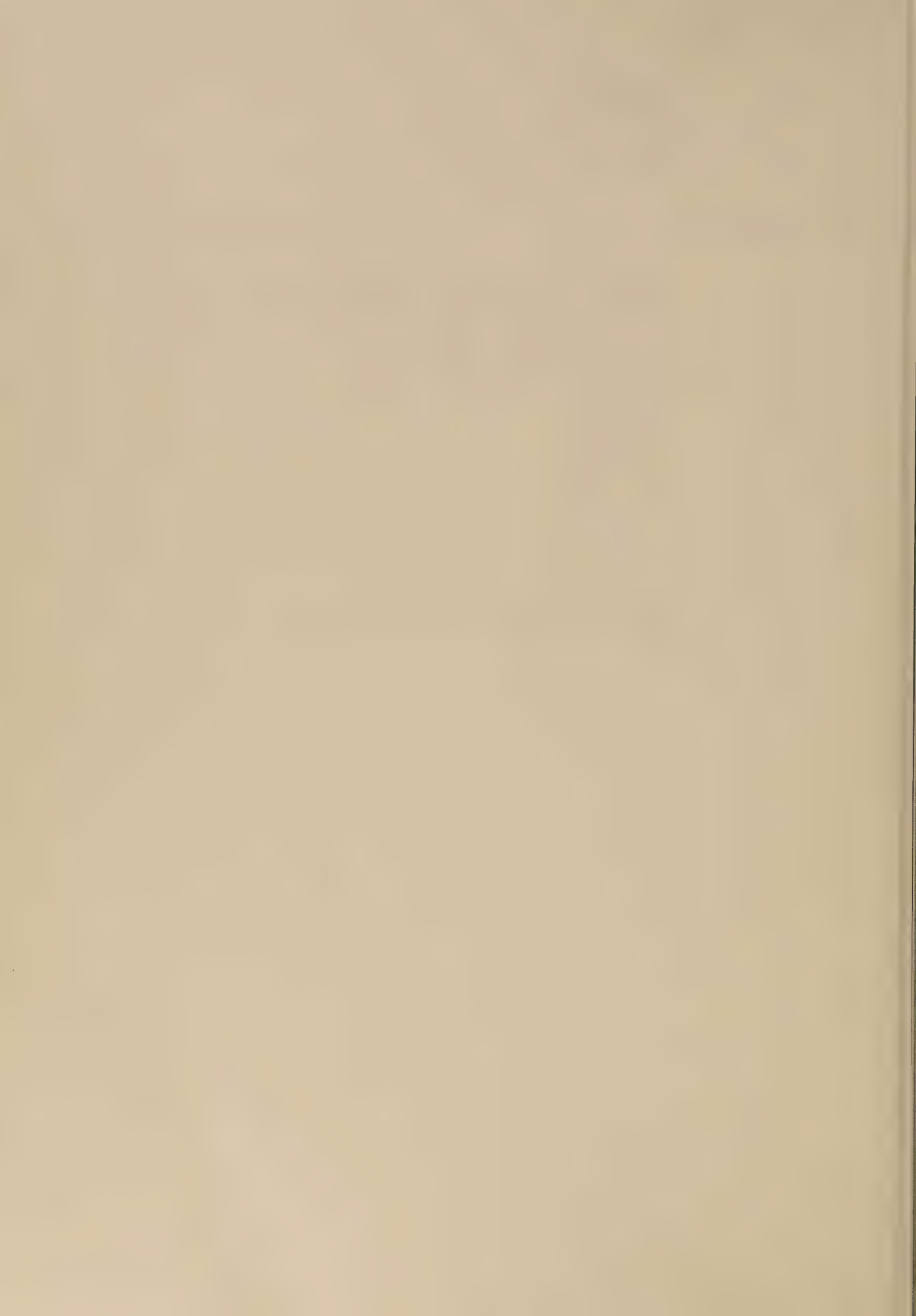
[†] not dimensionless in Appendix



+	additional
-	vector
=	tensor

Subscripts

i	grid-point index ("inner" field)
j	grid-point index ("inner" field)
k	grid-point index ("outer" field)
l	grid-point index ("outer" field)
x	x-direction
y	y-direction
z	z-direction
r	r-direction
η	η -direction
θ	θ -direction
ξ	ξ -direction
o	evaluated at the centre of a particle



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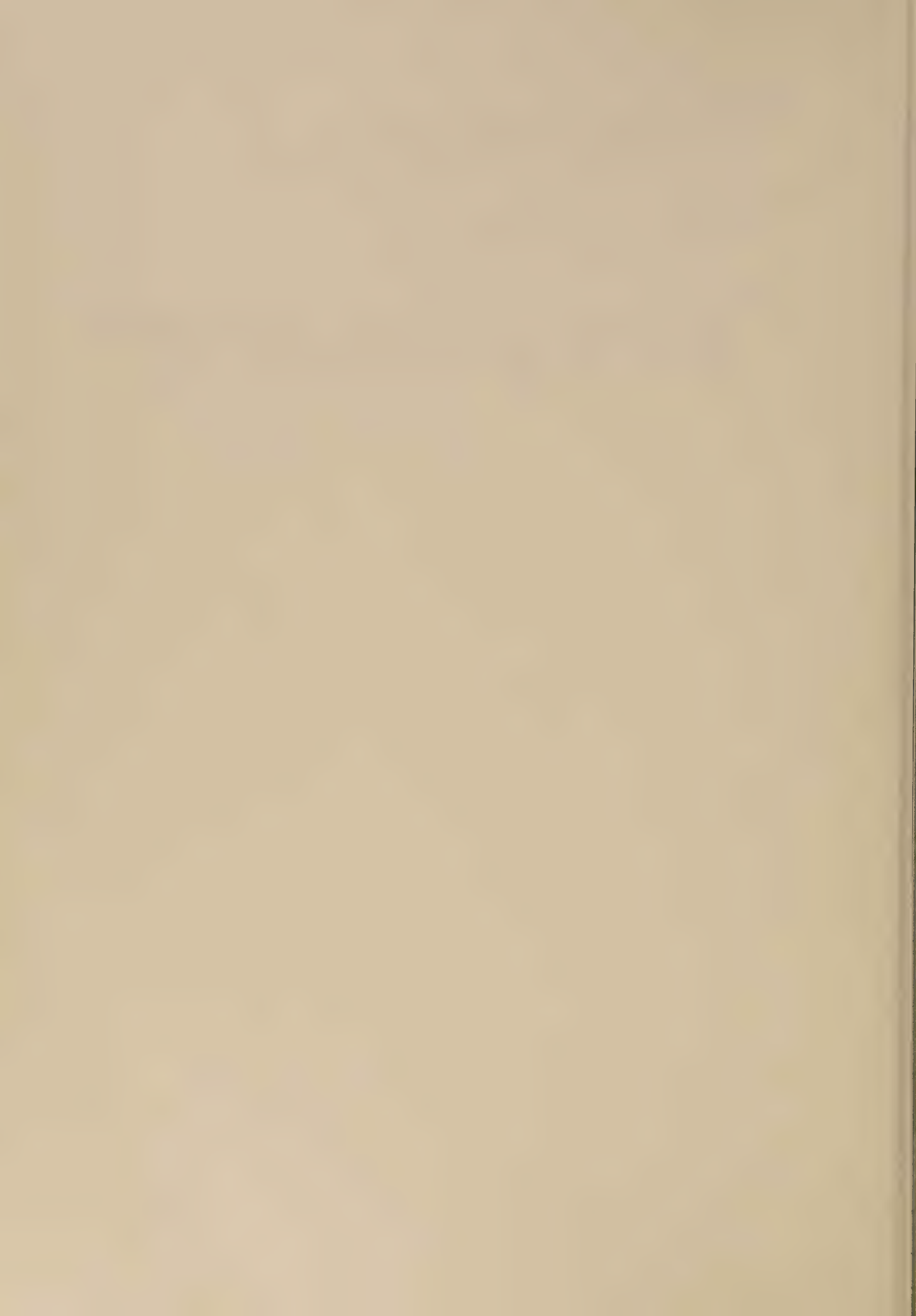
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APPENDIX 1

EQUATIONS

CONTENTS

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A2.2	EQUATIONS FOR THE PRESSURE	A1:5
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A1 INTRODUCTION

A numerous amount of equations are introduced in the report. This Appendix shows in detail how these equations are derived from the two basic equations of motion

The Navier-Stokes Equation

The Equation of Continuity

The equations are as far as possible derived to be applicable to a general two-dimensional problem. The equations can easily be extended to describe the flow around other body shapes, not parallell walls etc. However the finally derived equations in the Appendix are valid for the flow around a circular cylinder between to parallel plates.



A2 EQUATIONS

The basic equations in the calculations have been the Navier-Stokes equation

$$\nabla p + \rho \bar{\mathbf{v}} \cdot \nabla \bar{\mathbf{v}} - \mu \nabla^2 \bar{\mathbf{v}} = 0 \quad (\text{A1})$$

and the equation of continuity

$$\nabla \cdot \bar{\mathbf{v}} = 0 \quad (\text{A2})$$

These are the equations of motion for a steady, incompressible and viscous flow.

In the case of a two-dimensional flow the equations of motion can be simplified. If the streamfunction, ψ , is introduced the solution of the equations of motion is reduced to search for a single scalar function. Instead of solving a system of three equations, there is only one but fourth-order partial differential equation. It appears to be useful to introduce the vorticity, $\bar{\omega}$, which describes the rotation of the flow. For irrotational flow $\bar{\omega} = 0$. In terms of the streamfunction and the vorticity, the equations of motion reduce to a system of two scalar second-order partial differential equations.

The streamlines are defined by $\psi = \text{const.}$ and the isovorticity lines by $\omega = |\bar{\omega}| = \text{const.}$

A2.1 EQUATIONS FOR THE STREAMFUNCTION AND THE VORTICITY

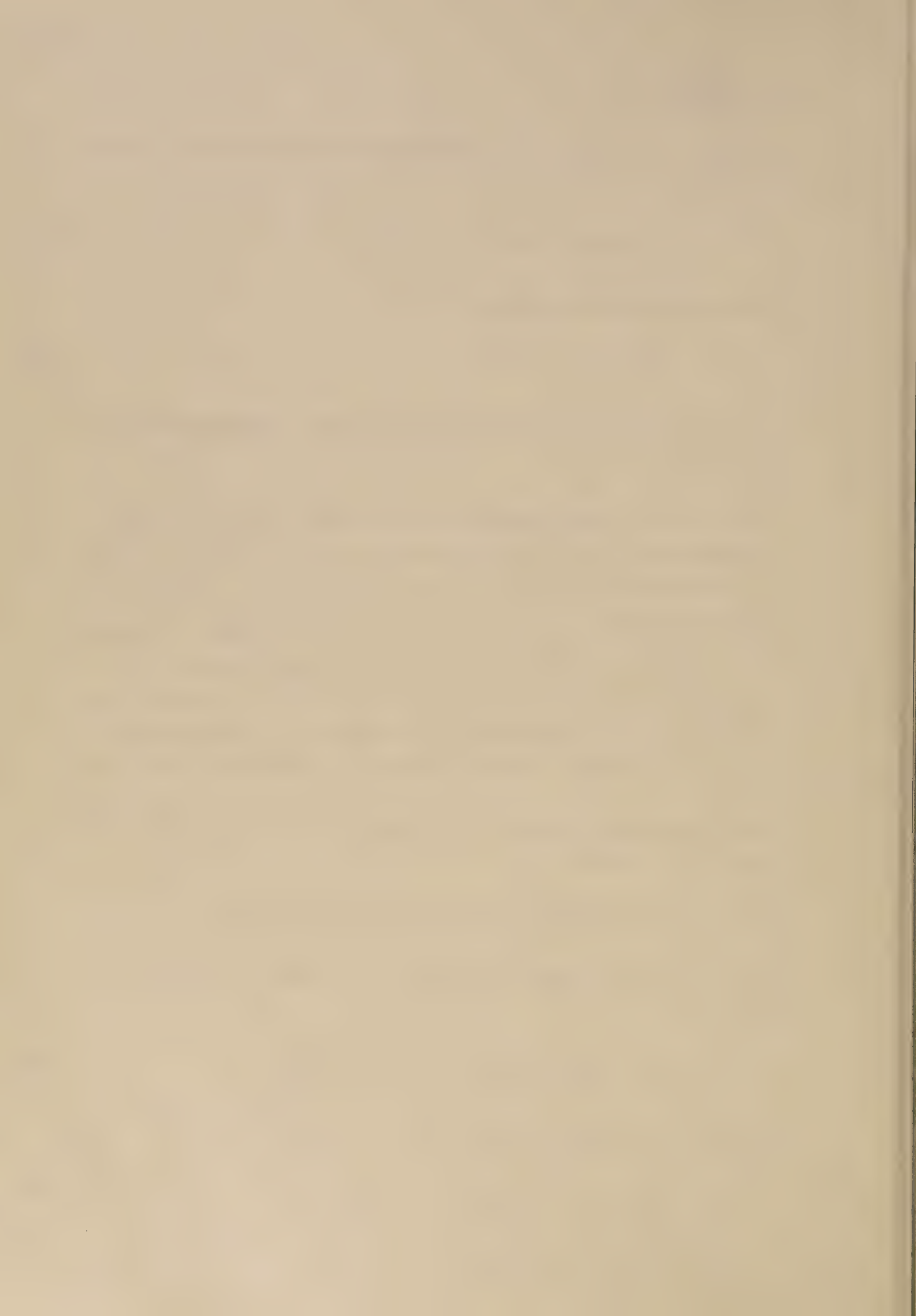
The Navier-Stokes equation (A1) can be expressed in an alternate form

$$\nabla \left(p + \frac{\rho \mathbf{v}^2}{2} \right) - \rho \bar{\mathbf{v}} \times (\nabla \times \bar{\mathbf{v}}) + \mu \nabla \times (\nabla \times \bar{\mathbf{v}}) = 0 \quad (\text{A3})$$

with use of the vector identities

$$\bar{\mathbf{v}} \cdot \nabla \bar{\mathbf{v}} = \frac{1}{2} \nabla \mathbf{v}^2 - \bar{\mathbf{v}} \times (\nabla \times \bar{\mathbf{v}}) \quad (\text{A4})$$

$$\nabla^2 \bar{\mathbf{v}} = \nabla (\nabla \cdot \bar{\mathbf{v}}) - \nabla \times (\nabla \times \bar{\mathbf{v}}) \quad (\text{A5})$$



and the equation of continuity

$$\nabla \cdot \bar{\mathbf{v}} = 0 \quad (\text{A2})$$

The vorticity, $\bar{\omega}$, is introduced defined according to

$$\bar{\omega} = \nabla \times \bar{\mathbf{v}} \quad (\text{A6})$$

A "vorticity equation" can then be obtained by taking the curl of eq (A3)

$$-\rho \nabla \times (\bar{\mathbf{v}} \times \bar{\omega}) + \mu \nabla \times (\nabla \times \bar{\omega}) = 0 \quad (\text{A7})$$

If the vector identity (A5) together with the relation

$$\nabla \cdot \bar{\omega} = \nabla \cdot (\nabla \times \bar{\mathbf{v}}) = 0 \quad (\text{A8})$$

is used, the final form of the "vorticity equation" becomes

$$\rho \nabla \times (\bar{\mathbf{v}} \times \bar{\omega}) + \mu \nabla^2 \bar{\omega} = 0 \quad (\text{A9})$$

For twodimensional flow (no z-dependence) the vorticity has only one component

$$\bar{\omega} = \bar{\mathbf{i}}_z |\bar{\omega}| = \bar{\mathbf{i}}_z \omega \quad (\text{A10})$$

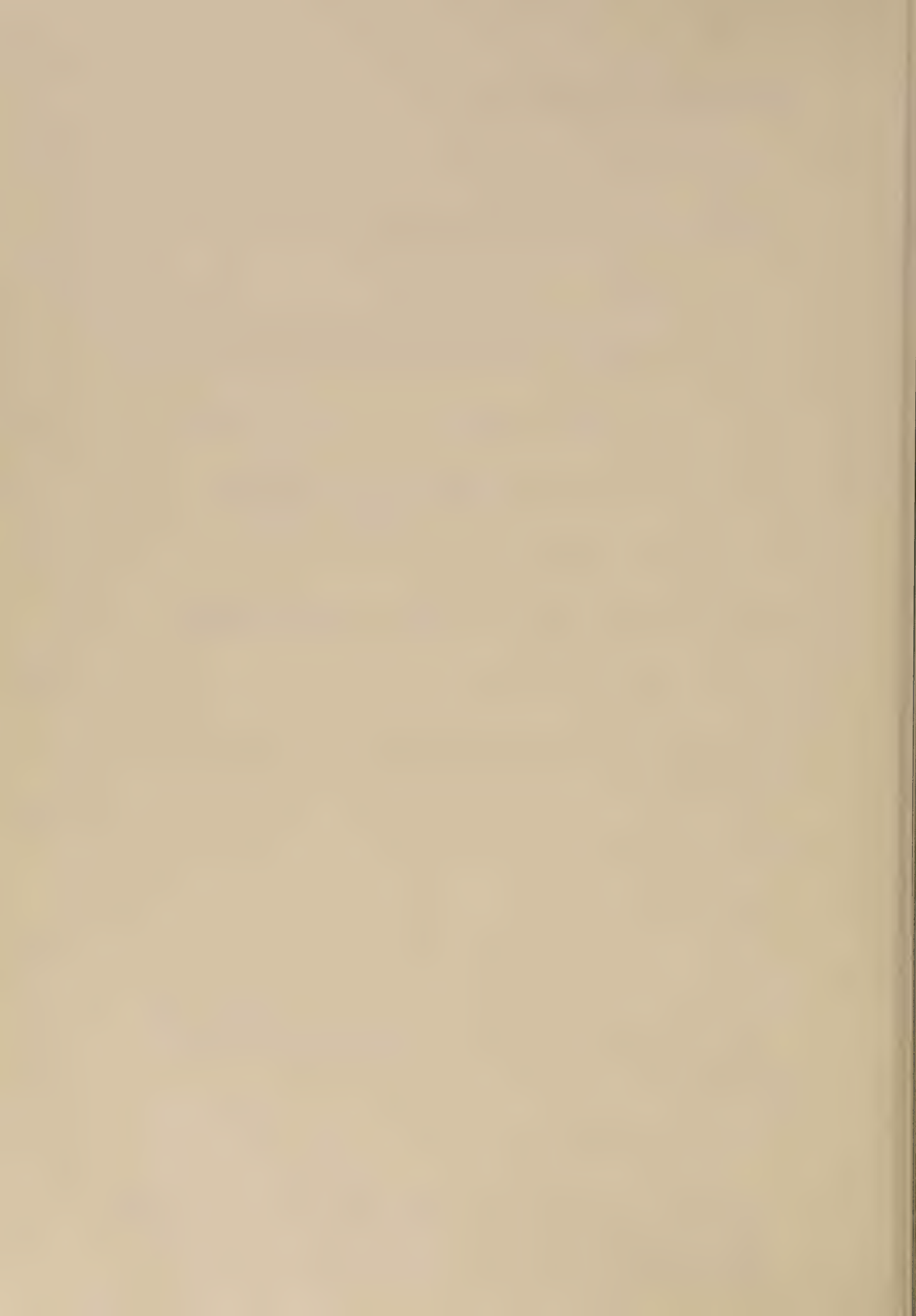
Eq (A9) then is

$$\rho \nabla \times (\bar{\mathbf{v}} \times \bar{\mathbf{i}}_z \omega) + \mu \bar{\mathbf{i}}_z \nabla^2 \omega = 0 \quad (\text{A11})$$

The equations of motion are thus eq (A11), (A6) and (A2). By introducing the streamfunction, ψ , related to the velocity, $\bar{\mathbf{v}}$, according to

$$\bar{\mathbf{v}} = \bar{\mathbf{i}}_z \times \nabla \psi \quad (\text{A12})$$

the number of equations are reduced to two, because the streamfunction automatically satisfies the equations of continuity. The streamfunction is related to the vorticity according to



$$\bar{\omega} = \bar{i}_z \omega = \nabla \times \bar{v} = \nabla \times (\bar{i}_z \times \nabla \psi) = \bar{i}_z \nabla^2 \psi \quad (A13)$$

and thus

$$\omega = \nabla^2 \psi \quad (A14)$$

The velocity in eq (A11) is eliminated with use of eq (A12)

$$\begin{aligned} \nabla \times (\bar{v} \times \bar{i}_z \omega) &= \nabla \times [(\bar{i}_z \times \nabla \psi) \times \bar{i}_z \omega] = \\ &= \nabla \times [\omega \nabla \psi - (\nabla \psi \cdot \bar{i}_z \omega) \bar{i}_z] = \\ &= \nabla \times \omega \nabla \psi = \\ &= \nabla \omega \times \nabla \psi + \omega (\nabla \times \nabla \psi) = \\ &= \nabla \omega \times \nabla \psi \end{aligned} \quad (A15)$$

where $\nabla \psi \cdot \bar{i}_z = 0$ because ψ is independent of z . The term $\nabla \omega \times \nabla \psi$ has a component only in the z -direction. Therefore, the equations of motion for two-dimensional flow (no z -dependence) in terms of the vorticity, ω , and the streamfunction, ψ , are expressed in these two scalar differential equations

$$\rho \bar{i}_z \cdot \nabla \omega \times \nabla \psi + \mu \nabla^2 \omega = 0 \quad (A16)$$

$$\nabla^2 \psi - \omega = 0 \quad (A14)$$

In rectangular coordinates eq (A16) and (A14) become

$$\rho \left(\frac{\partial \omega}{\partial x} \frac{\partial \psi}{\partial y} - \frac{\partial \omega}{\partial y} \frac{\partial \psi}{\partial x} \right) + \mu \nabla^2 \omega = 0 \quad (A17)$$

$$\nabla^2 \psi - \omega = 0 \quad (A18)$$

where ∇^2 is the Laplace operator

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad (A19)$$



The streamfunction and the vorticity are related to the velocity components according to

$$v_x = - \frac{\partial \psi}{\partial y} \quad (A20)$$

$$v_y = \frac{\partial \psi}{\partial x} \quad (A21)$$

$$\omega = \frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \quad (A22)$$

In circular cylindrical coordinates the corresponding equations are

$$\rho \left(\frac{1}{r} \frac{\partial \omega}{\partial r} \frac{\partial \psi}{\partial \theta} - \frac{1}{r} \frac{\partial \omega}{\partial \theta} \frac{\partial \psi}{\partial r} \right) + \mu \nabla^2 \omega = 0 \quad (A23)$$

$$\nabla^2 \psi - \omega = 0 \quad (A24)$$

where the operator, ∇^2 , is

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \quad (A25)$$

The relations between the velocity components and the streamfunction and vorticity are

$$v_r = - \frac{1}{r} \frac{\partial \psi}{\partial \theta} \quad (A26)$$

$$v_\theta = \frac{\partial \psi}{\partial r} \quad (A27)$$

and

$$\omega = \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r} \frac{\partial v_r}{\partial \theta} \quad (A28)$$

A2.2 EQUATIONS FOR THE PRESSURE

The pressure equations are also derived from the equations of motion (A1) and (A2). The Navier-Stokes equation is rewritten with use of the vector identity (A5) and the equation of continuity (A2)

$$\nabla p + \rho \vec{v} \cdot \nabla \vec{v} + \mu \nabla \times (\nabla \times \vec{v}) = 0 \quad (A29)$$

The pressure equation is then obtained by taking the divergence of eq (A29)

$$\nabla^2 p + \rho \nabla \cdot (\bar{\mathbf{v}} \cdot \nabla \bar{\mathbf{v}}) = 0 \quad (\text{A30})$$

The velocity, $\bar{\mathbf{v}}$, is obtained from the relation with ψ , eq (A12). ψ is obtained from eqs (A16) and (A14).

In rectangular coordinates eq (A24) is

$$\nabla^2 p + 2\rho \left[\left(\frac{\partial v_x}{\partial x} \right)^2 + \frac{\partial v_x}{\partial y} \frac{\partial v_y}{\partial x} \right] = 0 \quad (\text{A31})$$

where ∇^2 is the Laplace operator, eq (A19).

The velocity components are related to the streamfunction according to equations (A20) and (A21).

With circular cylindrical coordinates the pressure equation is

$$\nabla^2 p + 2\rho \left[\left(\frac{\partial v_r}{\partial r} \right)^2 + \frac{1}{r} \frac{\partial v_\theta}{\partial r} \left(\frac{\partial v_r}{\partial \theta} - v_\theta \right) \right] = 0 \quad (\text{A32})$$

where ∇^2 is the Laplace operator (A22). The velocity components are given by the expressions (A26) and (A27).

Pressure drop and pressure distributions are obtained by integration of the Navier-Stokes equation

$$\nabla p = \mu \nabla^2 \bar{\mathbf{v}} - \rho \bar{\mathbf{v}} \cdot \nabla \bar{\mathbf{v}} \quad (\text{A1})$$

or alternatively

$$\begin{aligned} \nabla \left(p + \frac{\rho \bar{\mathbf{v}}^2}{2} \right) &= \mu \nabla^2 \bar{\mathbf{v}} + \rho \bar{\mathbf{v}} \times \bar{\boldsymbol{\omega}} = \\ &= -\mu \nabla \times \bar{\boldsymbol{\omega}} + \rho \bar{\mathbf{v}} \times \bar{\boldsymbol{\omega}} \end{aligned} \quad (\text{A33})$$

because $\nabla \cdot \bar{\mathbf{v}} = 0$



A2.3 EQUATIONS FOR THE DRAG ON A BODY

The force on a body, exerted by the flow can be written

$$\bar{\mathbf{F}} = \iint_{\text{body}} \bar{\bar{\mathbf{\Pi}}} \cdot d\bar{\mathbf{S}} \quad (\text{A34})$$

where $\bar{\bar{\mathbf{\Pi}}}$ is the stress tensor

$$\bar{\bar{\mathbf{\Pi}}} = - p \bar{\mathbf{I}} + 2\mu \bar{\bar{\mathbf{\Delta}}} \quad (\text{A35})$$

in which $\bar{\mathbf{I}}$ is the idemfactor (unit tensor δ_{ij}) and $\bar{\bar{\mathbf{\Delta}}}$ the rate of deformation tensor

$$\bar{\bar{\mathbf{\Delta}}} = \frac{1}{2} [\nabla \bar{\mathbf{v}} + (\nabla \bar{\mathbf{v}})^T] \quad (\text{A36})$$

$d\bar{\mathbf{S}}$ is a directed element of surface area normal to the body and pointing into the surrounding fluid.

Thus if the fluid is incompressible and μ constant

$$\bar{\mathbf{F}} = - \iint_{\text{body}} p d\bar{\mathbf{S}} + 2\mu \iint_{\text{body}} \bar{\bar{\mathbf{\Delta}}} \cdot d\bar{\mathbf{S}} \quad (\text{A37})$$

The first term on the right side is the pressure force and the second the friction force.

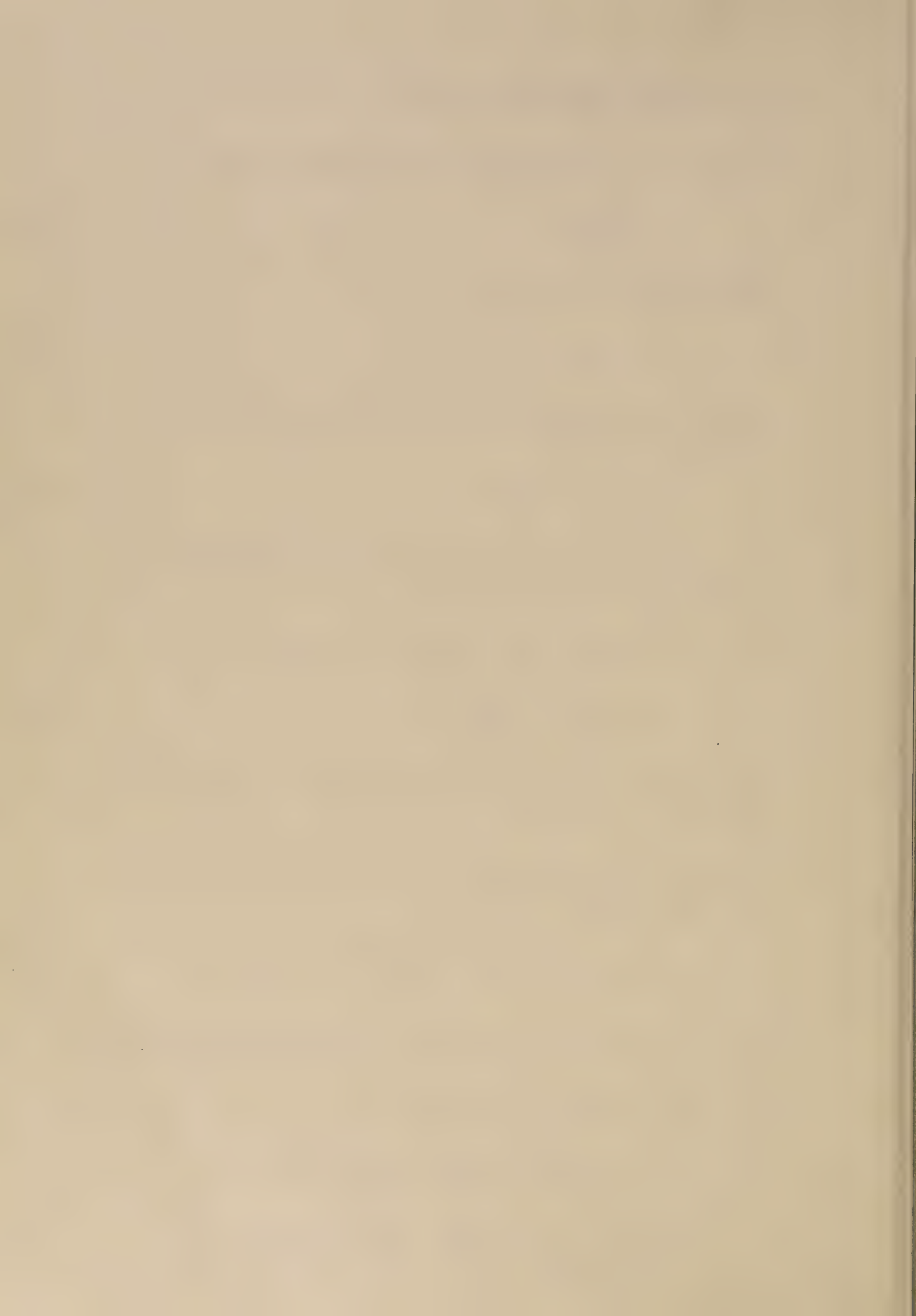
The drag on a circular cylinder per unit length in x-direction in a two-dimensional flow then is

$$F_{Dx} = - \bar{\mathbf{i}}_x \cdot \int_{S_5} p \bar{\mathbf{i}}_r dS + \bar{\mathbf{i}}_x \cdot 2\mu \int_{S_5} \bar{\bar{\mathbf{\Delta}}} \cdot \bar{\mathbf{i}}_r dS \quad (\text{A38})$$

where $dS = R d\theta$. See figure A1 for explanation of surface notations.

The rate of deformation tensor becomes

$$\begin{aligned} \bar{\bar{\mathbf{\Delta}}} = & \bar{\mathbf{i}}_r \bar{\mathbf{i}}_r \frac{\partial v_r}{r \partial r} + \bar{\mathbf{i}}_r \bar{\mathbf{i}}_\theta \frac{1}{2} \left[\frac{\partial v_\theta}{\partial r} + \frac{1}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} \right] + \\ & + \bar{\mathbf{i}}_\theta \bar{\mathbf{i}}_r \frac{1}{2} \left[\frac{\partial v_\theta}{\partial r} + \frac{1}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} \right] + \bar{\mathbf{i}}_\theta \bar{\mathbf{i}}_\theta \left[\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right] \end{aligned} \quad (\text{A39})$$



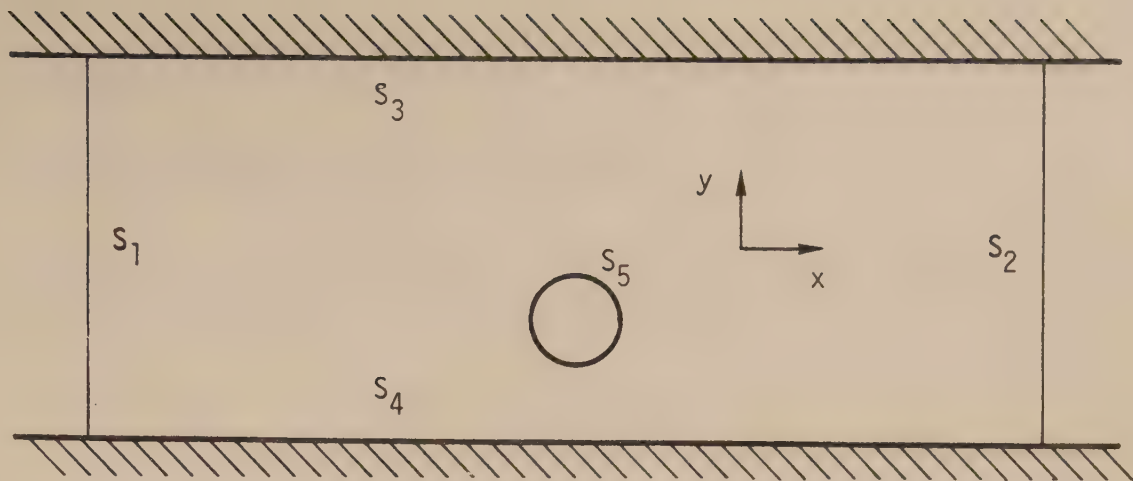


FIGURE A1. Surface notations.

Hence

$$\bar{\Delta} \cdot \bar{i}_r = \bar{i}_r \frac{\partial v_r}{\partial r} + \bar{i}_\theta \frac{1}{r} \left[\frac{\partial v_\theta}{\partial r} + \frac{1}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} \right] \quad (A40)$$

which with use of eq (A28) can be written

$$\bar{\Delta} \cdot \bar{i}_r = \bar{i}_r \frac{\partial v_r}{\partial r} + \bar{i}_\theta \frac{1}{r} \omega + \bar{i}_\theta \left[\frac{1}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r} \right] \quad (A41)$$

Together with the boundary conditions on the cylinder surface

$$\frac{\partial v_r}{\partial r} = \frac{\partial v_r}{\partial \theta} = v_\theta = 0 \quad (A42)$$

eq (A38) becomes

$$F_{Dx} = - \bar{i}_x \cdot \int_{S_5} p \bar{i}_r dS + \bar{i}_x \cdot 2\mu \int_{S_5} \frac{1}{r} \omega \bar{i}_\theta dS \quad (A43)$$

Furthermore

$$\bar{i}_x \cdot \bar{i}_r = \nabla x \cdot \bar{i}_r = \frac{\partial x}{\partial r} = \cos \theta \quad (A44)$$

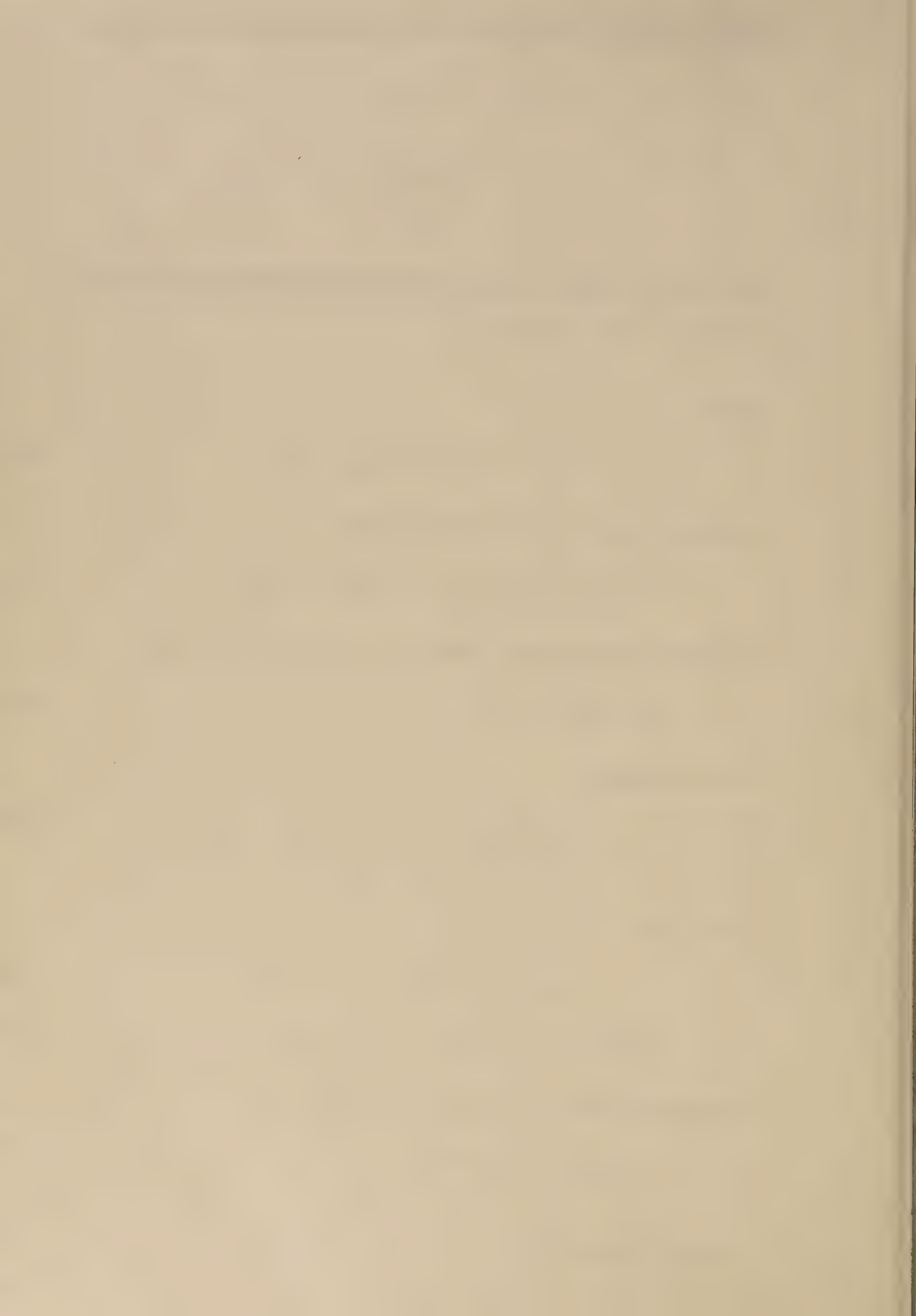
$$\bar{i}_x \cdot \bar{i}_\theta = \nabla x \cdot \bar{i}_\theta = \frac{1}{r} \frac{\partial x}{\partial \theta} = - \sin \theta \quad (A45)$$

The pressure drag is then calculated from

$$F_{Dpx} = - \int_0^{2\pi} p \cos \theta R d\theta \quad (A46)$$

and the friction drag

$$F_{Dfx} = - \mu \int_0^{2\pi} \omega \sin \theta R d\theta \quad (A47)$$



A2.4 EQUATIONS FOR THE FORCE ON THE WALLS

The force on the walls per unit length in x-direction can be calculated in a manner similar to that above

$$F_{wx} = - \bar{i}_x \cdot \int_{S_w} (p_3(-\bar{i}_y) + p_4 \bar{i}_y) dx + \bar{i}_x \cdot 2\mu \int_{S_w} (\bar{\Delta}_3 \cdot (-\bar{i}_y) + \bar{\Delta}_4 \cdot \bar{i}_y) dx \quad (A48)$$

where $S_w = S_3 = S_4$. The pressure terms vanish because $\bar{i}_x \cdot \bar{i}_y = 0$.

The rate of deformation tensor is

$$\bar{\Delta} = \bar{i}_x \bar{i}_x \frac{\partial v_x}{\partial x} + \bar{i}_x \bar{i}_y \frac{1}{2} \left[\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right] + \bar{i}_y \bar{i}_x \frac{1}{2} \left[\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right] + \bar{i}_y \bar{i}_y \frac{\partial v_y}{\partial y} \quad (A49)$$

and thus

$$\bar{\Delta} \cdot \bar{i}_y = \bar{i}_x \frac{1}{2} \left[\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right] + \bar{i}_y \frac{\partial v_y}{\partial y} \quad (A50)$$

In terms of the vorticity eq (A50) becomes

$$\bar{\Delta} \cdot \bar{i}_y = - \bar{i}_x \frac{1}{2} \omega + \bar{i}_x \frac{\partial v_y}{\partial x} + \bar{i}_y \frac{\partial v_y}{\partial y} \quad (A51)$$

The boundary conditions on the walls

$$\frac{\partial v_y}{\partial x} = \frac{\partial v_y}{\partial y} = 0 \quad (A52)$$

give

$$\bar{\Delta} \cdot \bar{i}_y = - \bar{i}_x \frac{1}{2} \omega \quad (A53)$$

Eq (48) now is

$$F_{wx} = \mu \int_{S_w} (\omega_3 - \omega_4) dx \quad (A54)$$



A2.5 MOMENTUM BALANCE

The Navier-Stokes equation

$$\nabla p + \rho \bar{\mathbf{v}} \cdot \nabla \bar{\mathbf{v}} - \mu \nabla^2 \bar{\mathbf{v}} = 0 \quad (\text{A1})$$

can be expressed in terms of the stress tensor, $\bar{\bar{\Pi}}$, eq (A35)

$$\nabla \cdot \bar{\bar{\Pi}} - \rho \nabla \cdot \bar{\mathbf{v}} \bar{\mathbf{v}} = 0 \quad (\text{A55})$$

where the equation of continuity, eq (A2), has been used.

The momentum balance for a flow within the surface S and with the volume V is

$$\iiint_V \nabla \cdot (\bar{\bar{\Pi}} - \rho \bar{\mathbf{v}} \bar{\mathbf{v}}) dV = 0 \quad (\text{A56})$$

Gauss divergence theorem gives rise to the expression

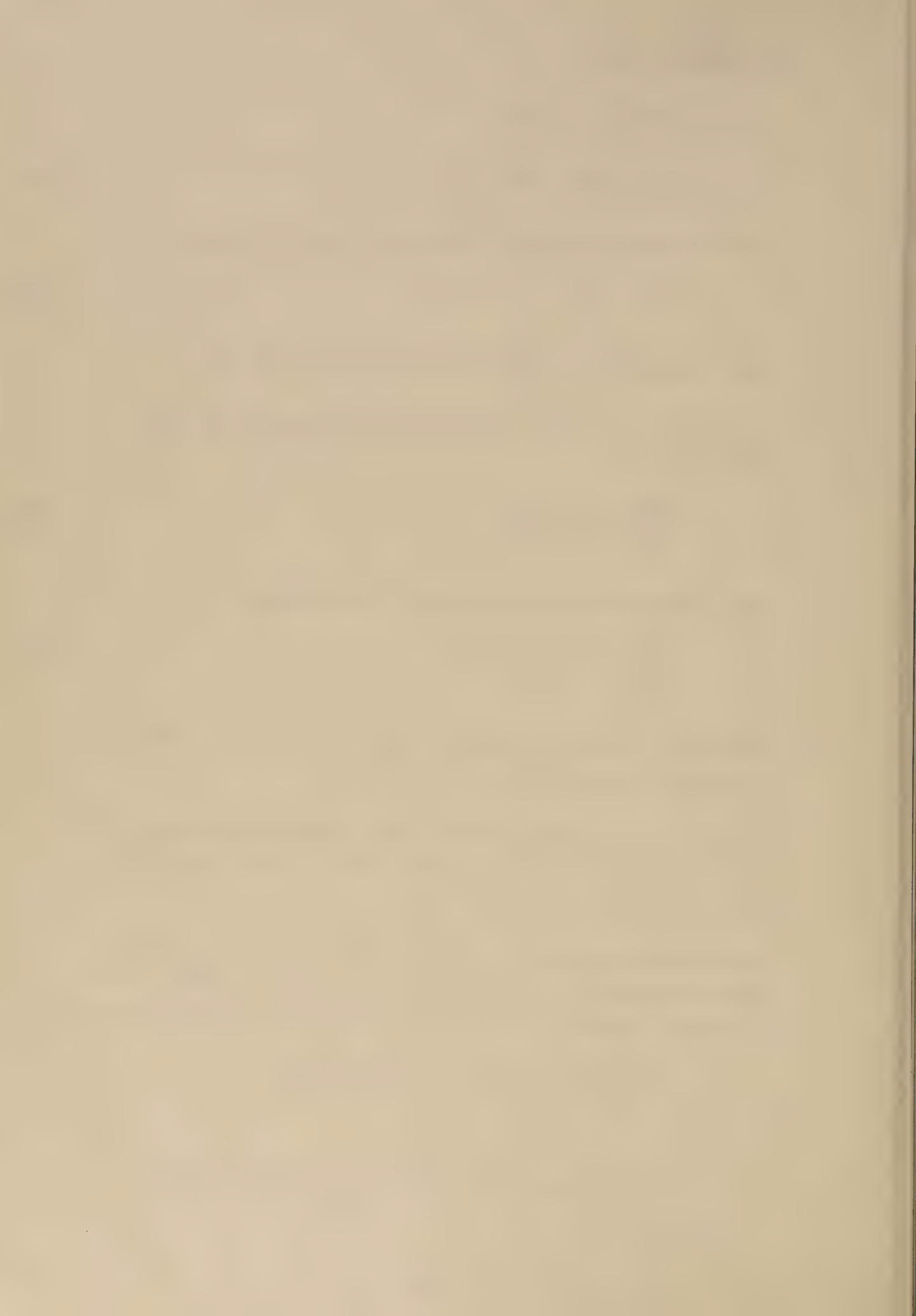
$$\iint_S (\bar{\bar{\Pi}} - \rho \bar{\mathbf{v}} \bar{\mathbf{v}}) \cdot d\bar{\mathbf{S}} = 0 \quad (\text{A57})$$

Where $d\bar{\mathbf{S}}$ is a directed element of surface area normal to the surface and pointing into the surrounding fluid

In the two-dimensional case with a body confined between parallel walls, the surface S can be divided into five parts (see fig A1), where $S_1 = S_2$ and $S_3 = S_4$.

On the solid surfaces $\bar{\mathbf{v}} = 0$. If the surfaces S_1 and S_2 are at a large distance from the body $\bar{v}_1 = \bar{v}_2$. The momentum balance, in the x -direction, applied to the system in fig A1 per unit length then is

$$\begin{aligned} & \bar{i}_x \cdot \int_{S_1} (-p\bar{\bar{I}}) \cdot \bar{i}_x dy + \bar{i}_x \cdot \int_{S_2} (-p\bar{\bar{I}}) \cdot (-\bar{i}_x) dy + \\ & + \bar{i}_x \cdot \int_{S_3} (-p\bar{\bar{I}} + 2\mu\bar{\bar{\Delta}}) \cdot (-\bar{i}_y) dx + \bar{i}_x \cdot \int_{S_4} (-p\bar{\bar{I}} + 2\mu\bar{\bar{\Delta}}) \cdot \bar{i}_y dx + \\ & + \bar{i}_x \cdot \int_{S_5} (-p\bar{\bar{I}} + 2\mu\bar{\bar{\Delta}}) \cdot \bar{i}_r R d\theta = 0 \end{aligned} \quad (\text{A58})$$



The third and fourth terms can be recognized as F_{wx} and the fifth term as F_{Dx} . If the pressure is constant in the cross section at S_1 and S_2 eq (A58) reduces to

$$\Delta p L - F_{wx} - F_{Dx} = 0 \quad (A59)$$

where

$$\Delta p = p_1 - p_2 \quad (A60)$$

is the pressure drop. L is the distance between the plates.

For the unperturbated flow field (no obstacle present) the momentum balance is

$$\Delta p^0 L - F_{wx}^0 = 0 \quad (A61)$$

and hence

$$\Delta p^+ L - F_{wx}^+ - F_{Dx} = 0 \quad (A62)$$

Δp^+ and F_{wx}^+ are the additional pressure drop and wall force caused by the presence of the body

$$\Delta p^+ = \Delta p - \Delta p^0 \quad (A63)$$

$$F_{wx}^+ = F_{wx} - F_{wx}^0 \quad (A64)$$

A2.6 RELATIONSHIPS BETWEEN THE DRAG ON THE BODY, THE ADDITIONAL PRESSURE DROP AND THE ADDITIONAL WALL FORCE

The rate of mechanical energy dissipation per unit time in a volume V of fluid is

$$E = \iiint_V \phi dV \quad (A65)$$

where ϕ is the local rate of energy dissipation per unit time and unit volume.



For creeping motion (Stokes' régime, $Re = 0$)

$$\Phi = \nabla \cdot (\bar{\mathbf{v}} \cdot \bar{\bar{\Pi}}) \quad (\text{A66})$$

where $\bar{\bar{\Pi}}$ is the stress tensor defined in eq (A35). Substituting eq (A66) in eq (A65) and applying Gauss' divergence theorem gives

$$E = - \iint_S \bar{\mathbf{v}} \cdot \bar{\bar{\Pi}} \cdot d\bar{\mathbf{S}} \quad (\text{A67})$$

where S is a closed surface bounding the volume V , and $d\bar{\mathbf{S}}$ is a directed element of surface area normal to the surface and pointing into the surrounding fluid.

The rate of mechanical energy dissipation per unit width and unit time for the system in fig A1, with the boundary conditions $\bar{\mathbf{v}} = 0$ on the solid surfaces and $\bar{\mathbf{v}} = \bar{\mathbf{v}}^0 = v^0 \bar{\mathbf{i}}_x$ (= the unperturbated velocity) on the surfaces S_1 and S_2 , is

$$E = \Delta p U L \quad (\text{A68})$$

where U is the mean velocity for the undisturbed flow. The rate of energy dissipation for the unperturbated flow field (absence of particle) is

$$E^0 = \Delta p^0 U L \quad (\text{A69})$$

and hence the additional rate of energy dissipation per unit width is

$$E^+ = \Delta p^+ U L \quad (\text{A70})$$

where

$$\Delta p^+ = \Delta p - \Delta p^0 \quad (\text{A63})$$

Brenner [67] has shown that the additional rate of energy dissipation also can be calculated as



$$E^+ = \bar{v}_0^0 \cdot \bar{F}_D \quad (A71)$$

when the body is small compared to the conduit. $\bar{v}_0^0$ is the approach velocity of the unperturbed velocity field evaluated at the centre of the particle. For Poiseuille flow around a cylinder situated half-way between two parallel plates $\bar{v}_0^0 = \frac{3}{2}U\bar{i}_x$ and $\bar{F}_D = F_{Dx}\bar{i}_x$ and thus

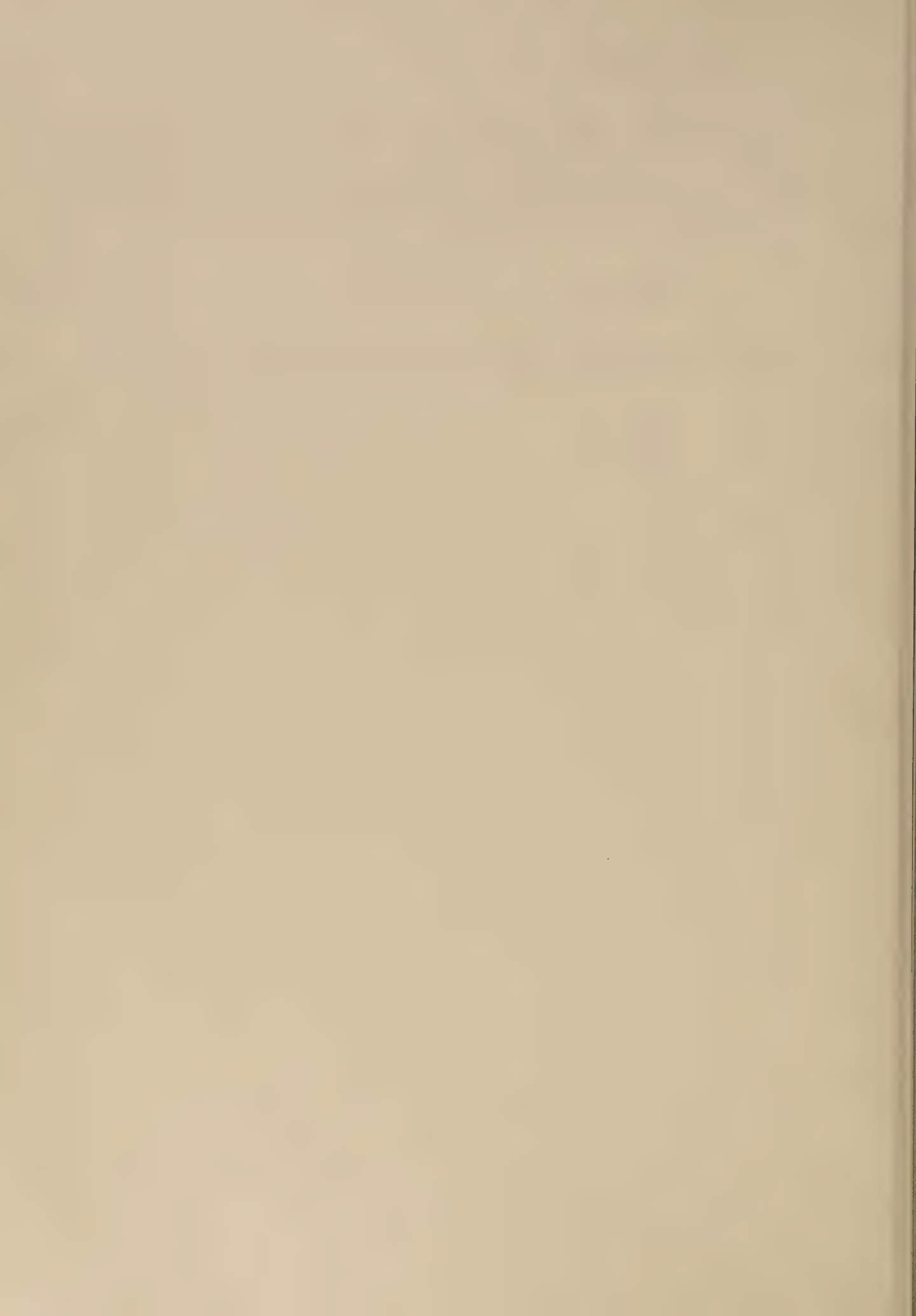
$$\frac{\Delta p^+ L}{F_{Dx}} = \frac{3}{2} \quad (A72)$$

From eq (A62) further relationships can be obtained

$$\frac{F_{wx}^+}{F_{Dx}} = \frac{1}{2} \quad (A73)$$

and

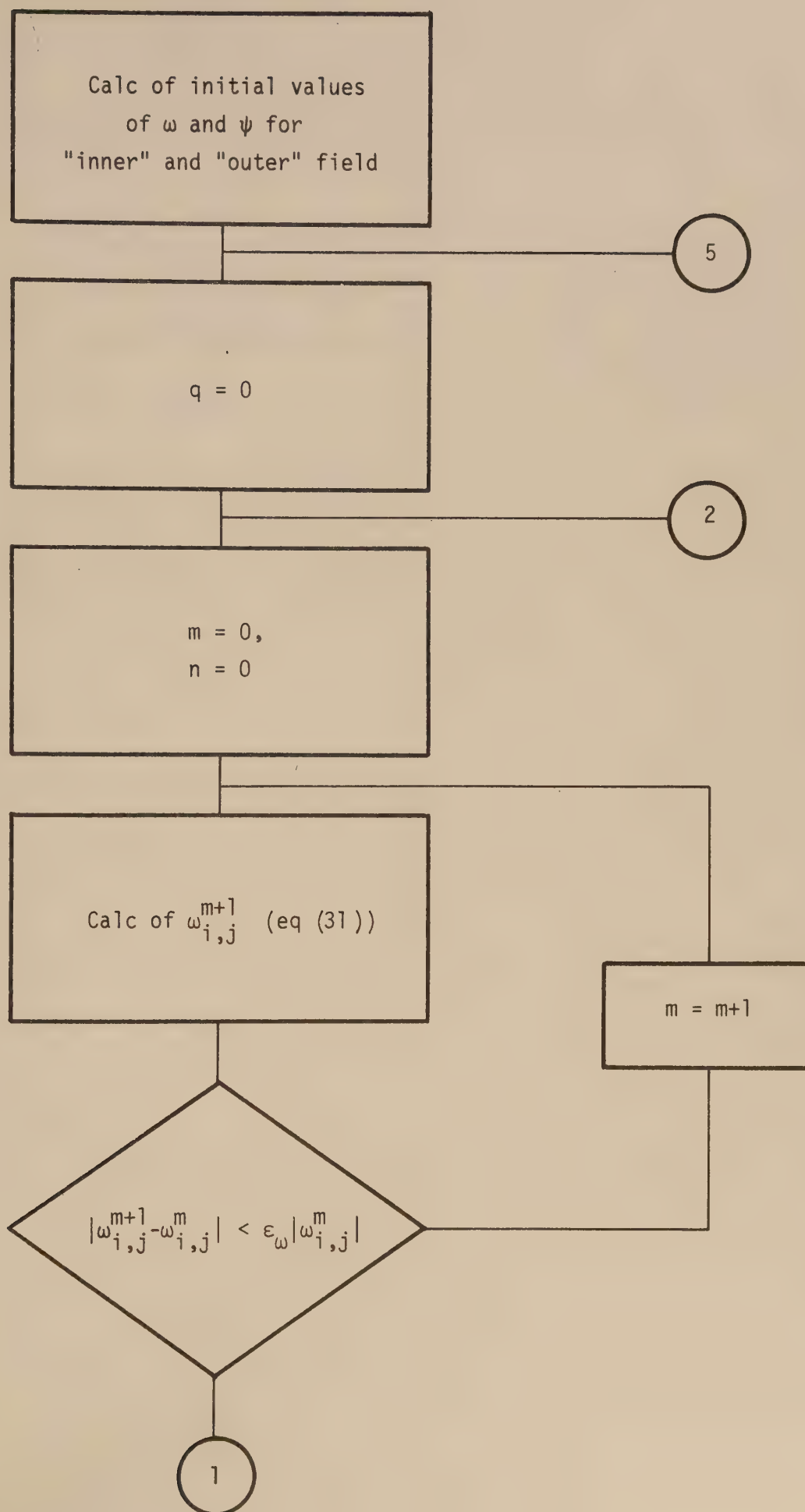
$$\Delta p^+ L = 3F_{wx}^+ \quad (A74)$$

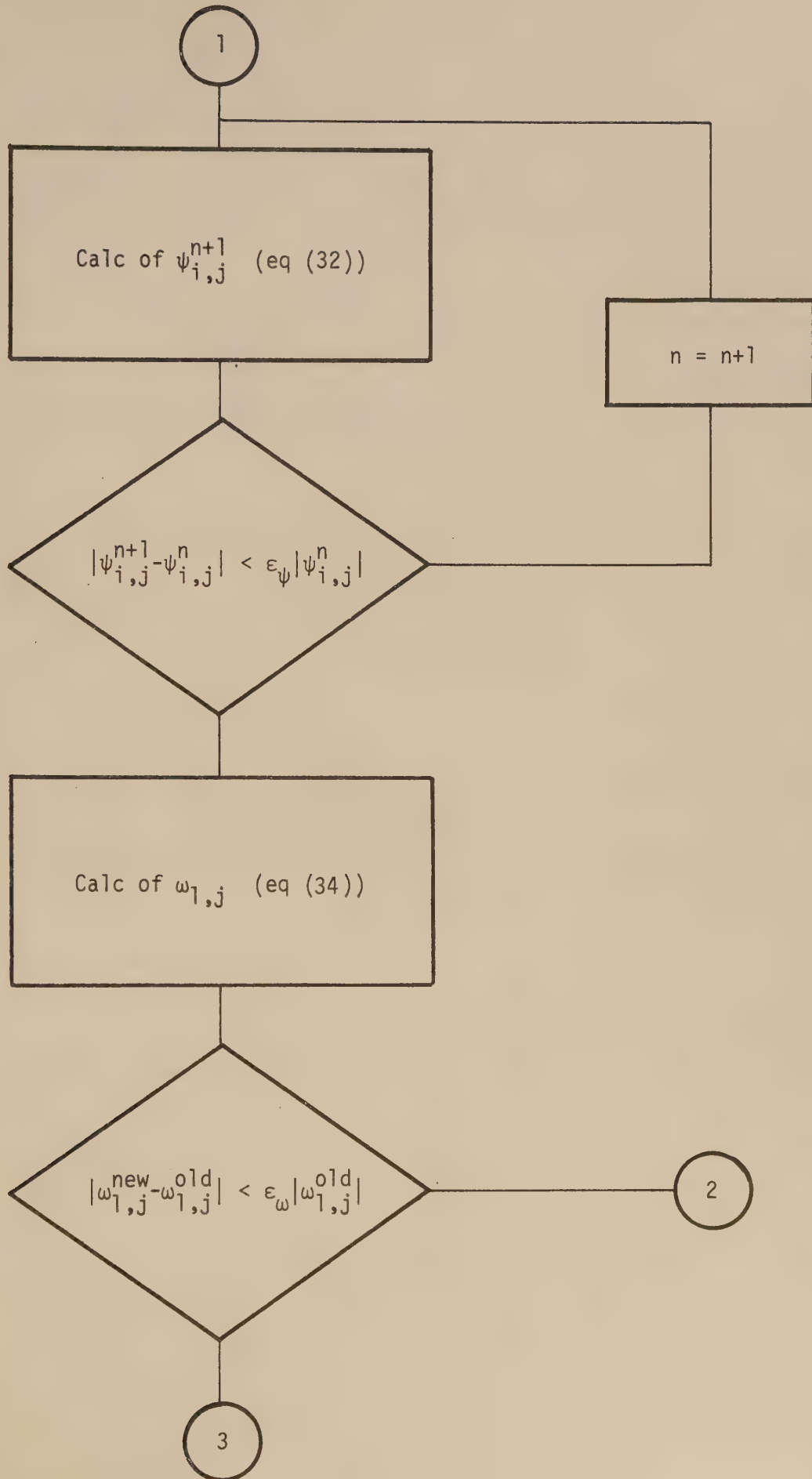


APPENDIX 2

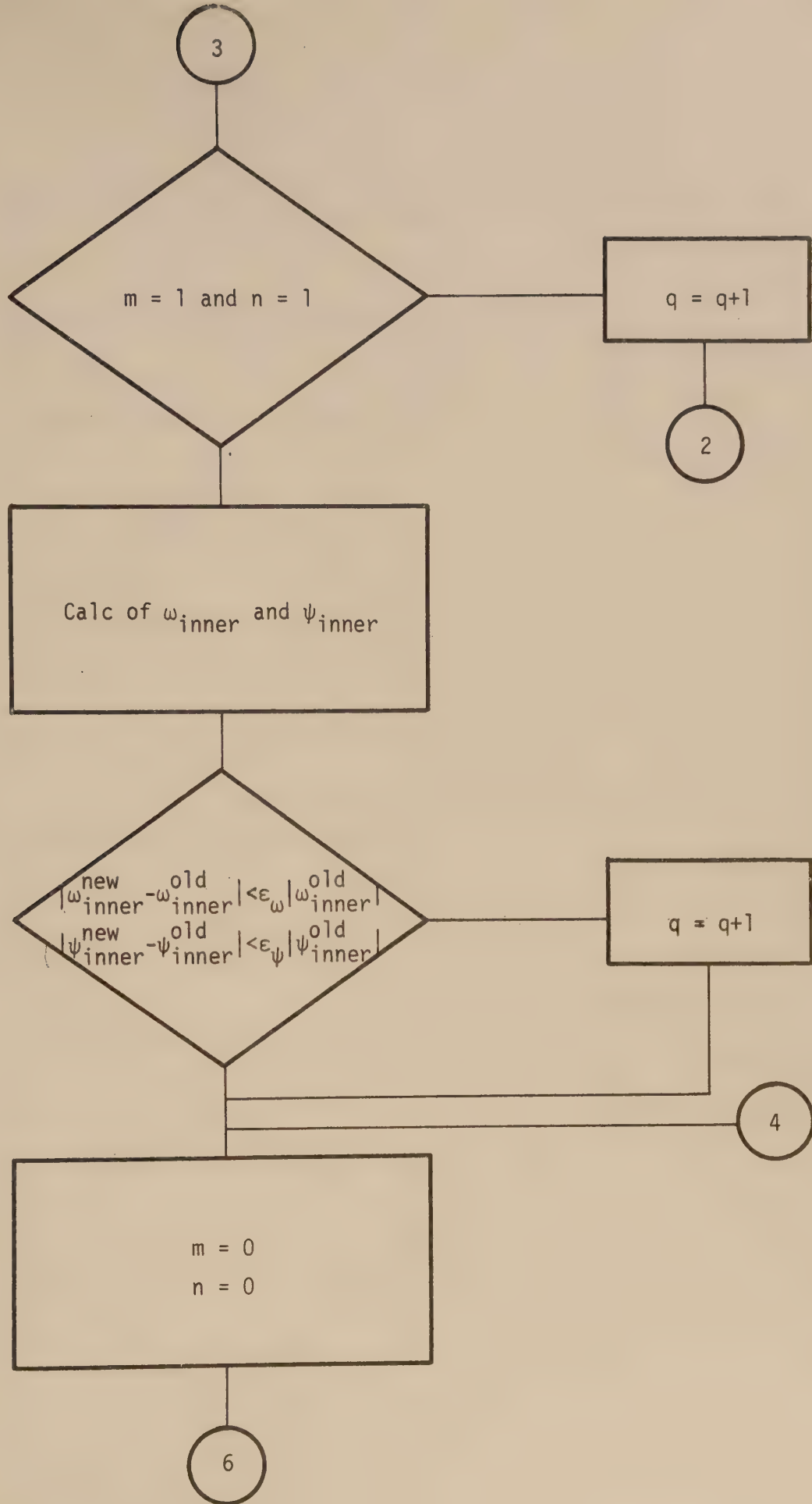
FLOW SHEET FOR THE VORTICITY AND
THE STREAMFUNCTION CALCULATIONS

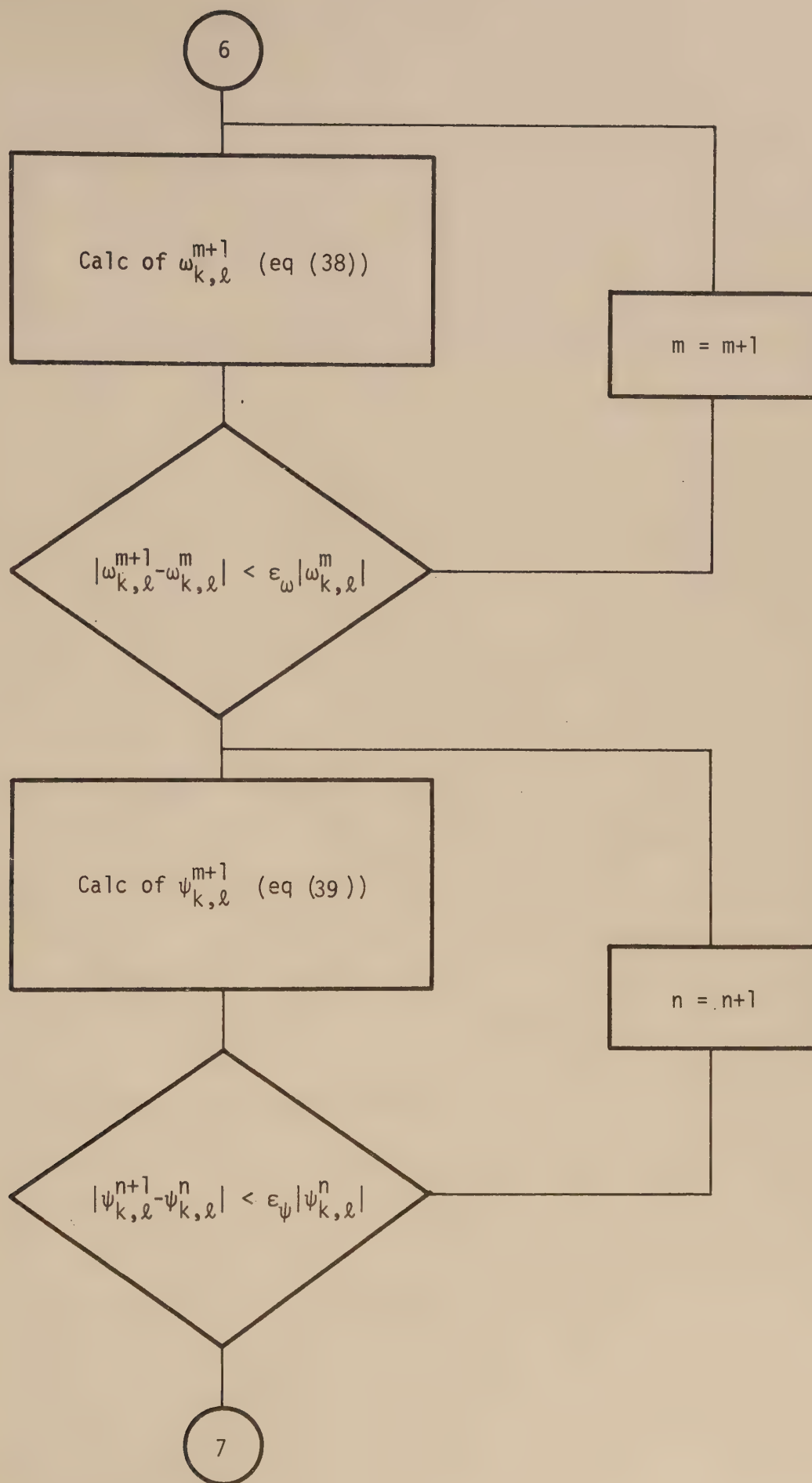


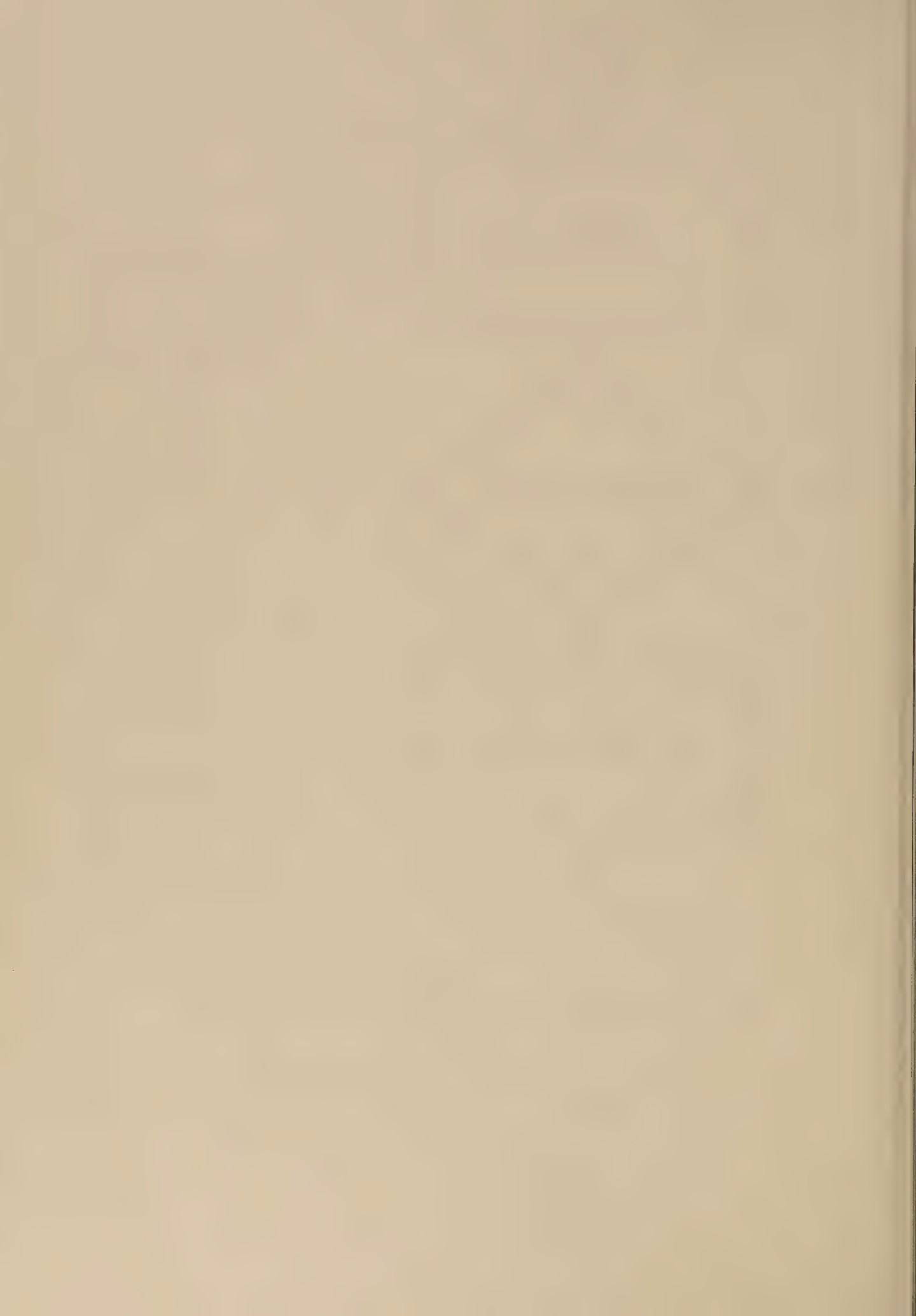


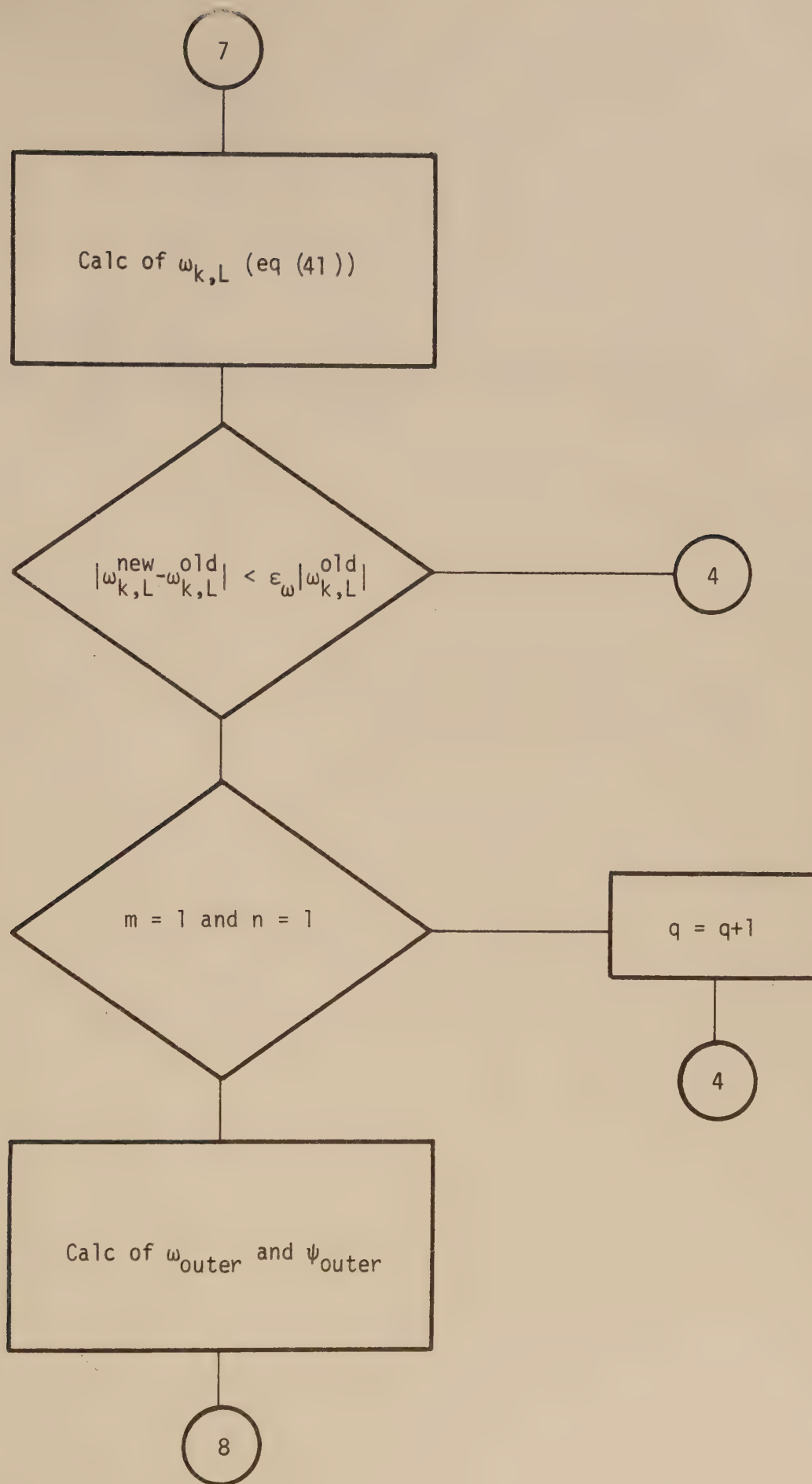


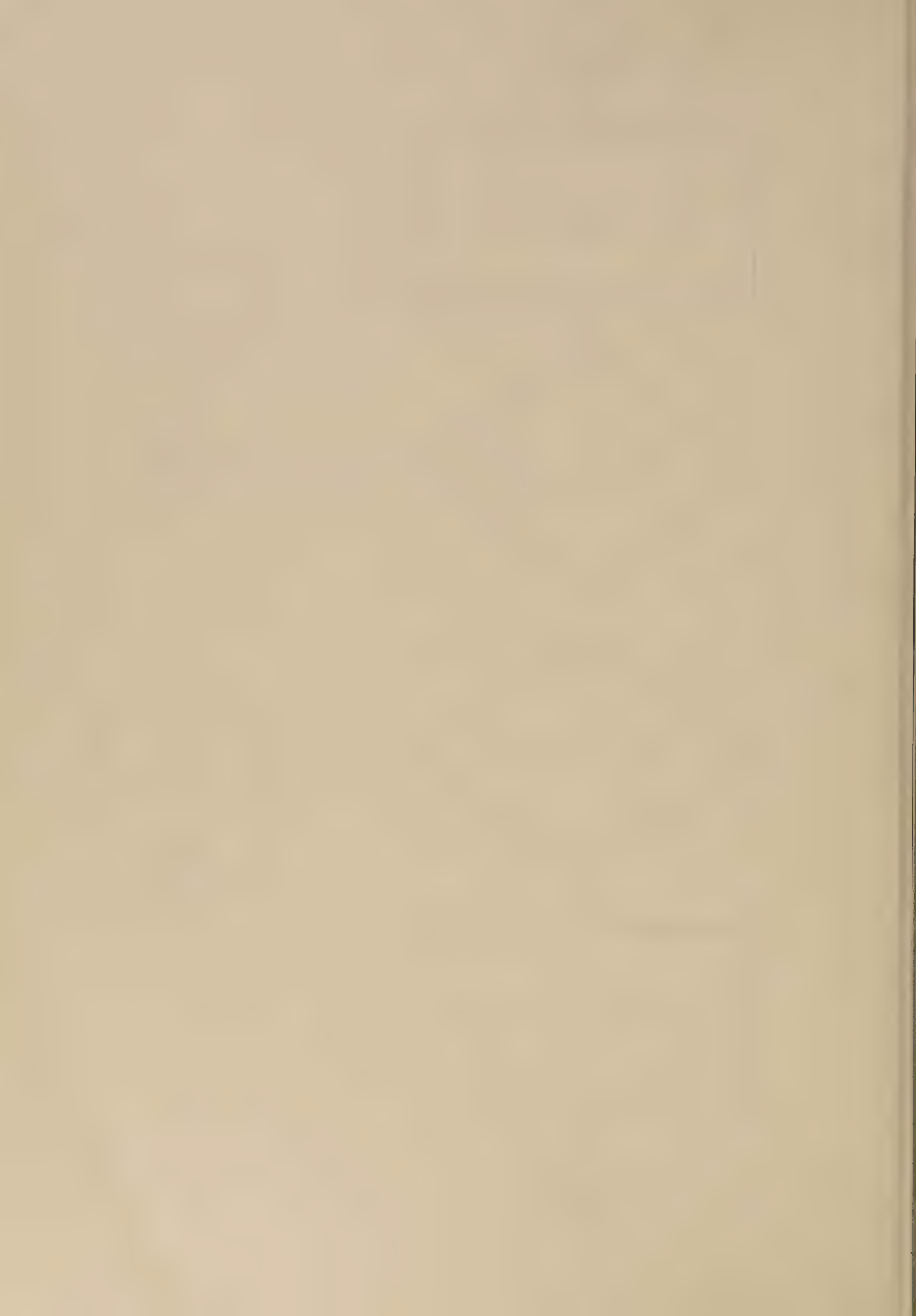


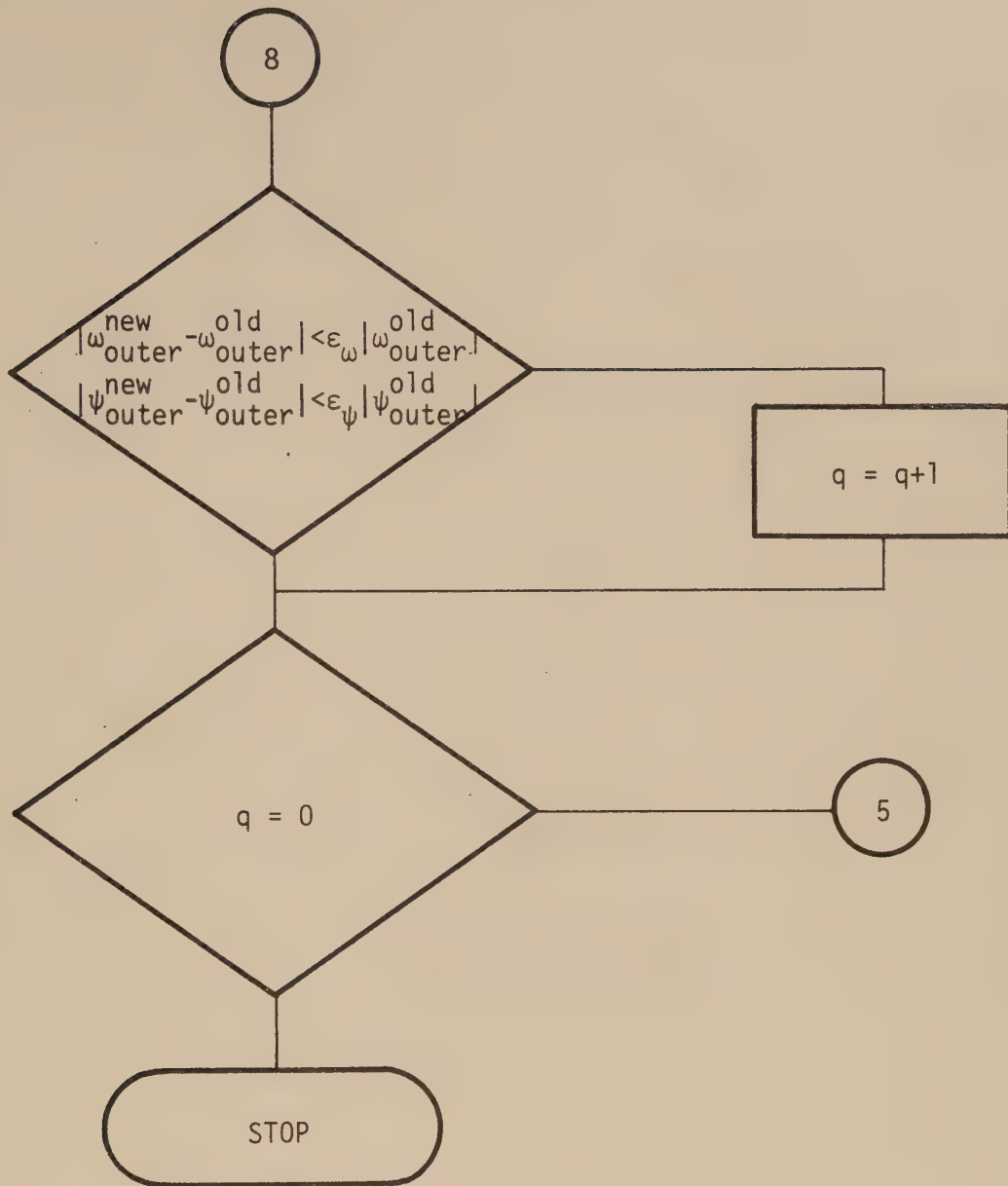










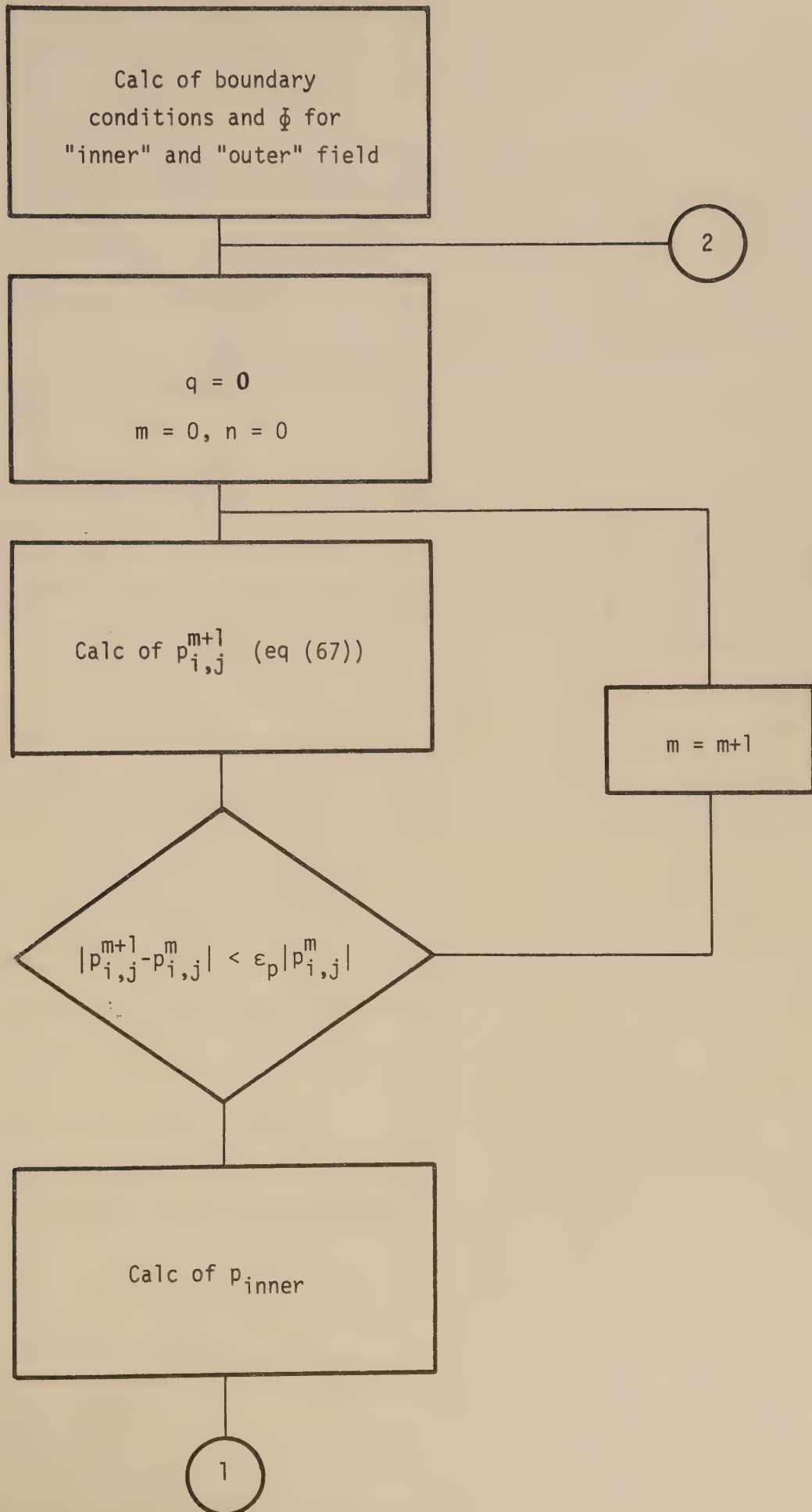




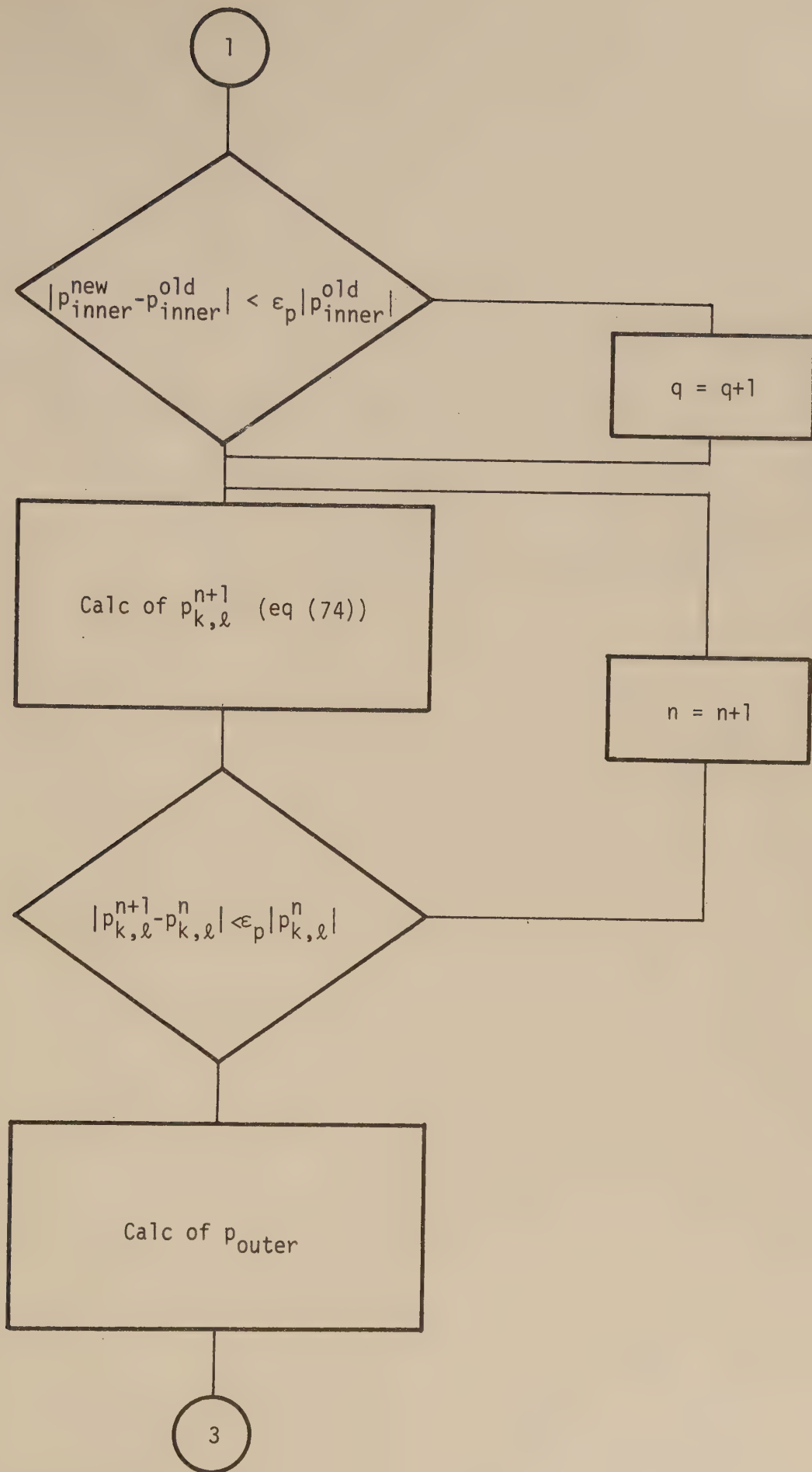
APPENDIX 3

FLOW SHEET FOR THE
PRESSURE CALCULATIONS

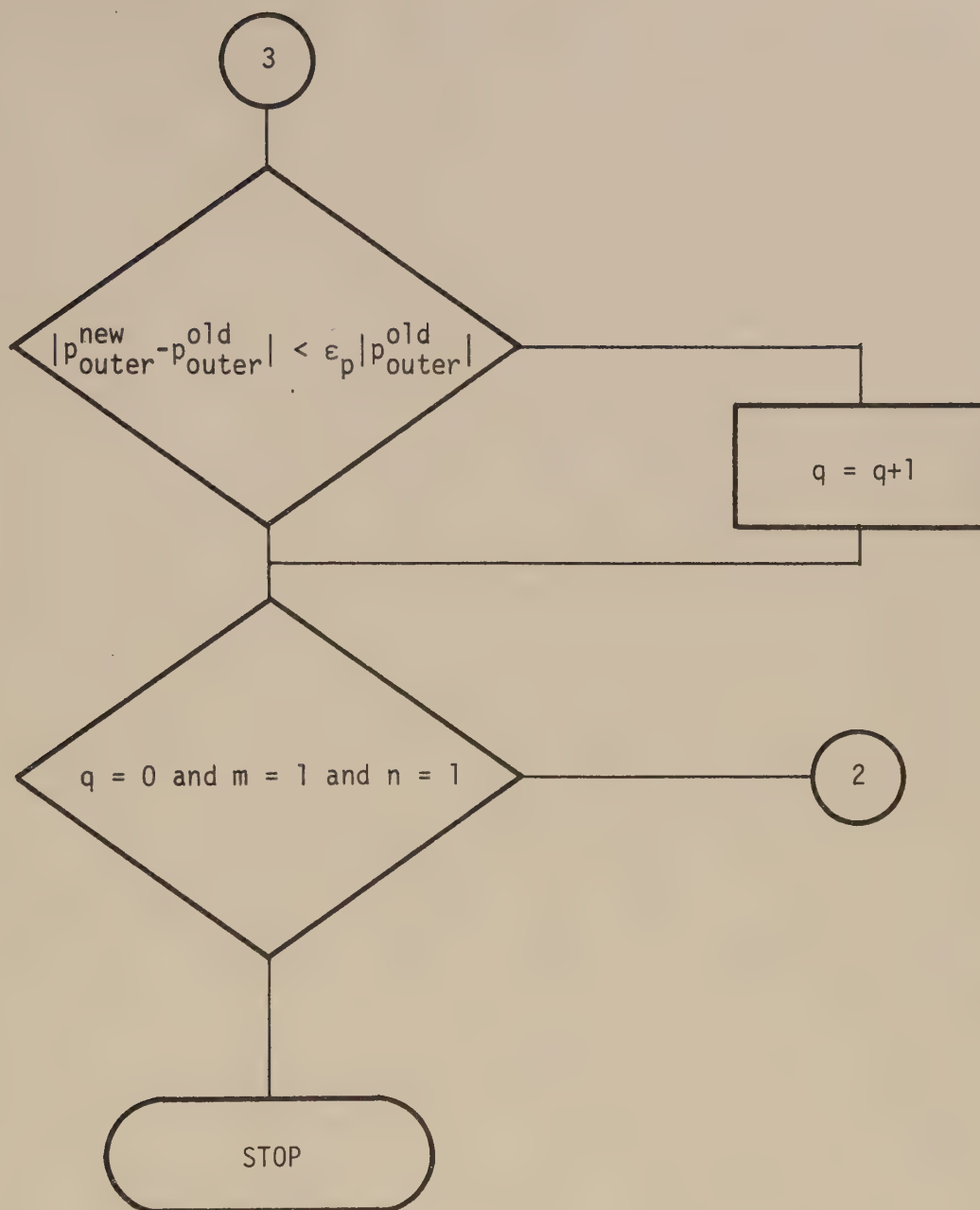


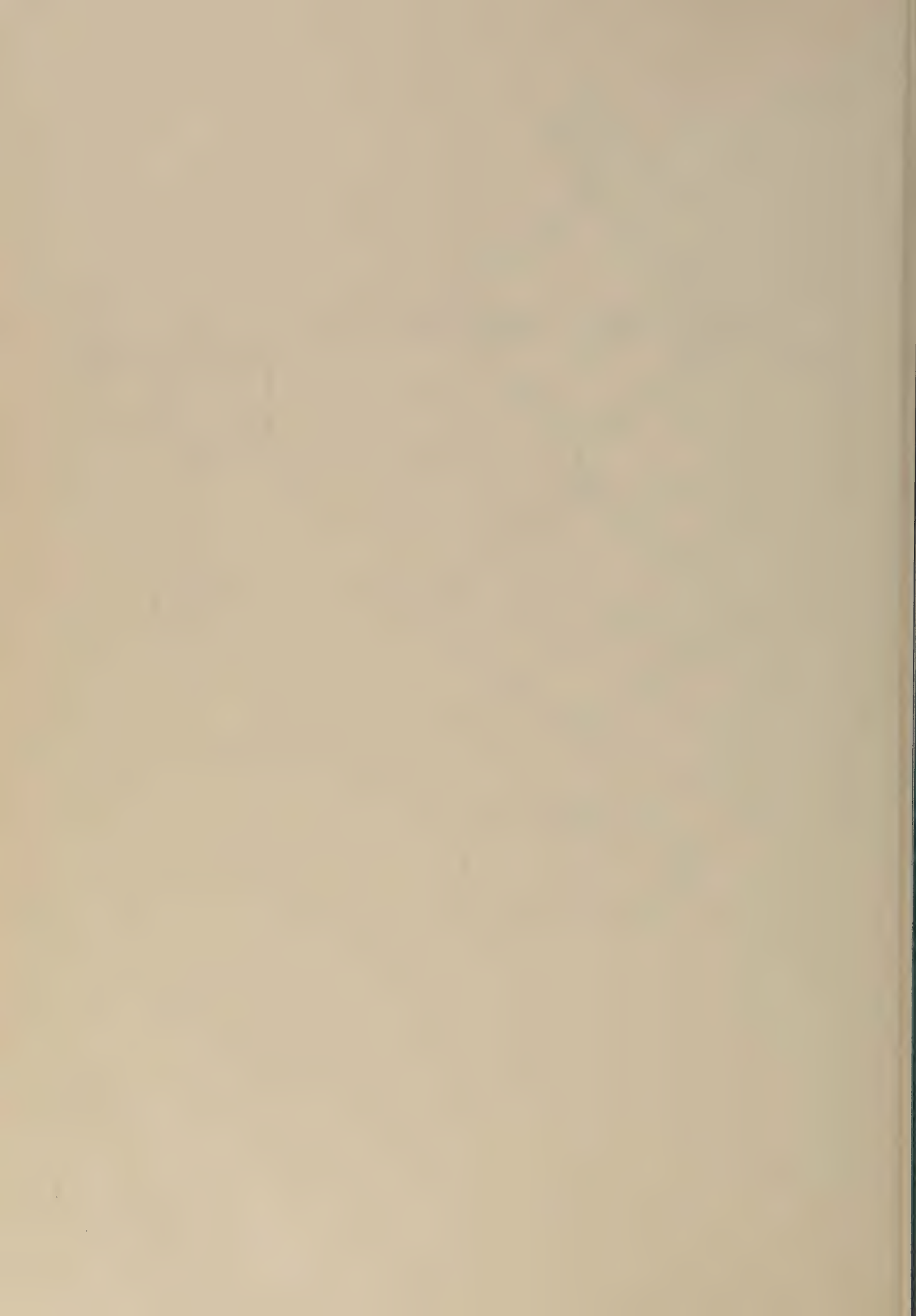












A NUMERICAL SOLUTION OF THE
FLOW AROUND A SPHERE IN A
CIRCULAR CYLINDER

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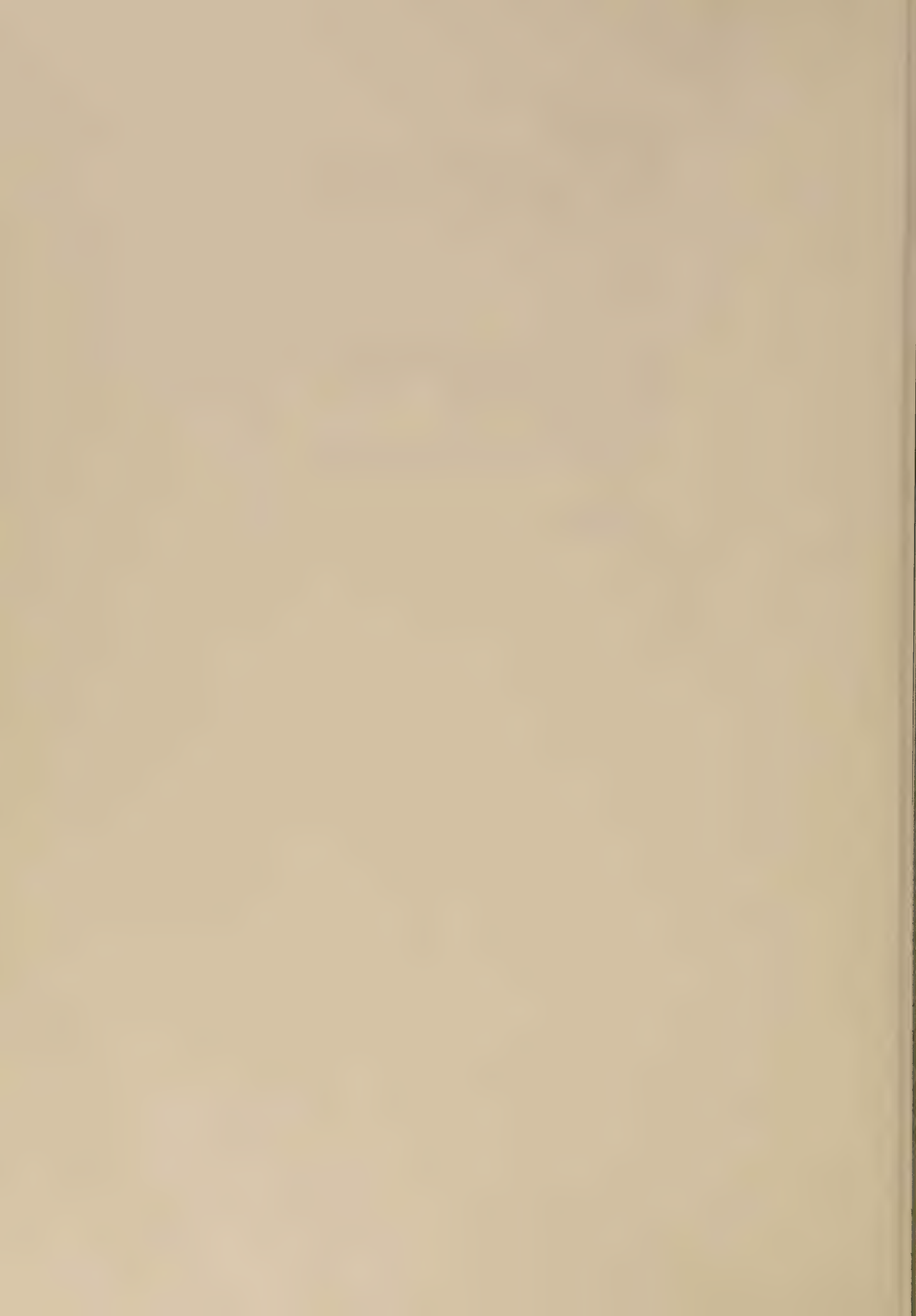
REPORT NO 74-F-3

ABSTRACT

A numerical solution of the Navier-Stokes equation is presented for Poiseuille flow around an axially placed, fixed sphere in a circular cylinder. Streamlines and isovorticity lines are calculated from the governing equations for the streamfunction and the vorticity. Isobars are calculated from a Poisson equation, derived from the Navier-Stokes equation. The pressure and vorticity distributions on the surface of the sphere, the additional pressure drop and the drag coefficients are presented. Solutions are obtained for Reynolds numbers up to 150 (based on cylinder diameter and mean velocity). The wall effects are examined by comparison with results of previous investigations for an unbounded flow around a sphere.

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1 INTRODUCTION

This paper presents a numerical solution of the Navier-Stokes equation for Poiseuille flow around a centrally placed, fixed sphere in a circular cylinder for different Reynolds numbers. For low Reynolds numbers it is possible to simplify the Navier-Stokes equation to a linear or quasi-linear differential equation. This problem has been dealt with i.a. by Faxén [1], Wakiya [2], Happel and Byrne [3], Haberman and Sayre [4], Bohlin [5] and Happel and Brenner [6]. The present paper deals with higher Reynolds numbers - up to 150 (based on tube diameter and mean velocity) - and the complete Navier-Stokes equation is used. Solutions are obtained by a finite difference method. In order to examine the wall effects, comparison is made with results for an unbounded flow around a sphere; Le Clair et al. [7] and Dennis and Walker [8].

2 EQUATIONS

The basic equations are the Navier-Stokes equation and the equation of continuity. The flow is assumed to be steady and incompressible.

$$\nabla p + \rho \bar{\mathbf{v}} \cdot \nabla \bar{\mathbf{v}} - \mu \nabla^2 \bar{\mathbf{v}} = 0 \quad (1)$$

$$\nabla \cdot \bar{\mathbf{v}} = 0 \quad (2)$$

The flow is also supposed to be axisymmetric and thus the velocity field can be expressed in terms of the well known scalar differential equations for the streamfunction, ψ , and the vorticity, ω . The equation for the pressure field is obtained in a Poisson equation, derived from the Navier-Stokes equation

$$\nabla^2 p + \rho \nabla \cdot (\bar{\mathbf{v}} \cdot \nabla \bar{\mathbf{v}}) = 0 \quad (3)$$

In order to obtain proper boundary conditions for the arising difference equations, the computational field is divi-



ded into two parts; the "inner" field and the "outer" field, which are overlapping to give a "common" field; see fig 1. In the "inner" field the equations are written in terms of the coordinates (ξ, θ) , where $\xi = \ln(r/r_0)$, r_0 is the radius of the sphere and (r, θ) are the polar coordinates in a plane through the axis of the cylinder. The transformation of the radial polar coordinate, r , has been made since it is desirable to have a finer mesh close to the surface of the sphere, where gradients are large. In the "outer" field the equations are expressed in the cylindrical coordinates (ρ, z) . The flow is in direction $\theta = 0$.

In the following all equations are made dimensionless; lengths with respect to the tube radius, R , velocity components with respect to the mean velocity in the undisturbed Poiseuille flow, U , and the pressure with respect to the dynamic head $\frac{1}{2} \rho U^2$. The Reynolds number, Re , is defined as $Re = \frac{2RU\rho}{\mu}$.

3 STREAMLINES AND ISOVORTICITY LINES

3.1 "INNER" FIELD

The equations for the streamfunction, ψ , and the vorticity, ω , in terms of the coordinates (ξ, θ) become

$$E^2 \Omega + \frac{Re}{2} \frac{\sin \theta}{r_0 e^\xi} \left[\frac{\partial \psi}{\partial \theta} \frac{\partial}{\partial \xi} \left(\frac{\Omega}{(r_0 e^\xi \sin \theta)^2} \right) - \frac{\partial \psi}{\partial \xi} \frac{\partial}{\partial \theta} \left(\frac{\Omega}{(r_0 e^\xi \sin \theta)^2} \right) \right] = 0 \quad (4)$$

$$E^2 \psi - \Omega = 0 \quad (5)$$

where Ω is defined by

$$\Omega = \omega r_0 e^\xi \sin \theta \quad (6)$$



E^2 is the operator

$$E^2 = \frac{1}{r_0^2 e^{2\xi}} \left[\frac{\partial^2}{\partial \xi^2} - \frac{\partial}{\partial \xi} + \sin \theta \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \right) \right] \quad (7)$$

The streamfunction and the vorticity are related to the velocity components according to

$$v_r = - \frac{1}{r_0^2 e^{2\xi} \sin \theta} \frac{\partial \psi}{\partial \theta} ; v_\theta = \frac{1}{r_0^2 e^{2\xi} \sin \theta} \frac{\partial \psi}{\partial \xi} \quad (8); (9)$$

$$\omega = \frac{1}{r_0 e^\xi} \left[\frac{\partial v_\theta}{\partial \xi} + v_\theta - \frac{\partial v_r}{\partial \theta} \right] \quad (10)$$

The boundary conditions for the "inner" field are

1) Along the axis of symmetry ($\theta = 0$ and $\theta = \pi$)

$$\psi = 0 \text{ and } \Omega = 0$$

2) On the surface of the sphere ($\xi = 0$)

$$\psi = 0 \text{ and } \Omega = \frac{1}{r_0^2} \frac{\partial^2 \psi}{\partial \xi^2}$$

3) On the "outer" boundary

$$\psi = \psi_{\text{outer}} \text{ and } \Omega = \Omega_{\text{outer}}$$

3.2 "OUTER" FIELD

For the "outer" field the corresponding equations are

$$E^2 \Omega + \frac{\text{Re}}{2} \frac{1}{\rho} \left[\frac{\partial \psi}{\partial \rho} \frac{\partial \Omega}{\partial z} - \frac{\partial \psi}{\partial z} \left(\frac{\partial \Omega}{\partial \rho} - 2 \frac{\Omega}{\rho} \right) \right] = 0 \quad (11)$$

$$E^2 \psi - \Omega = 0 \quad (12)$$



where Ω and E^2 are

$$\Omega = \omega \rho \quad (13)$$

$$E^2 = \frac{\partial^2}{\partial \rho^2} - \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} \quad (14)$$

The relations between the velocity components and the stream-function and the vorticity are

$$v_z = -\frac{1}{\rho} \frac{\partial \psi}{\partial \rho} ; v_\rho = \frac{1}{\rho} \frac{\partial \psi}{\partial z} \quad (15); (16)$$

and

$$\omega = \frac{\partial v_\rho}{\partial z} - \frac{\partial v_z}{\partial \rho} \quad (17)$$

The boundary conditions for the "outer" field are

1) Along the axis of symmetry ($\rho = 0$)

$$\psi = 0 \text{ and } \Omega = 0$$

2) Far away from the sphere ($z \rightarrow \pm\infty$)

$$\psi = 0.5 \rho^2 (\rho^2 - 2) \text{ and } \Omega = 4 \rho^2$$

3) On the cylinder wall ($\rho = 1$)

$$\psi = -0.5 \text{ and } \Omega = \frac{\partial^2 \psi}{\partial \rho^2}$$

4) On the "inner" boundary

$$\psi = \psi_{\text{inner}} \text{ and } \Omega = \Omega_{\text{inner}}$$



4 ISOBARS

4.1 "INNER" FIELD

The dimensionless pressure equation for the "inner" field in terms of the velocity components becomes

$$\nabla^2 p - \frac{4}{r_0^2 e^{2\xi}} \left[\frac{\partial v_r}{\partial \xi} \left(\frac{\partial v_\theta}{\partial \theta} + v_r \right) - \frac{\partial v_\theta}{\partial \xi} \left(\frac{\partial v_r}{\partial \theta} - v_\theta \right) - (v_r + \cot \theta v_\theta)^2 \right] = 0 \quad (18)$$

where the operator ∇^2 is

$$\nabla^2 = \frac{1}{r_0^2 e^{2\xi}} \left[\frac{1}{e^\xi} \frac{\partial}{\partial \xi} \left(e^\xi \frac{\partial}{\partial \xi} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) \right] \quad (19)$$

The velocity components are related to the streamfunction according to equations (8) and (9).

The pressure on the surface of the sphere at $\theta = \pi/2$ is chosen as a reference pressure, p_{ref} .

The pressure distribution on the boundaries of the "inner" field is given by:

1) Along the axis of symmetry ($\theta = 0$ and $\theta = \pi$)

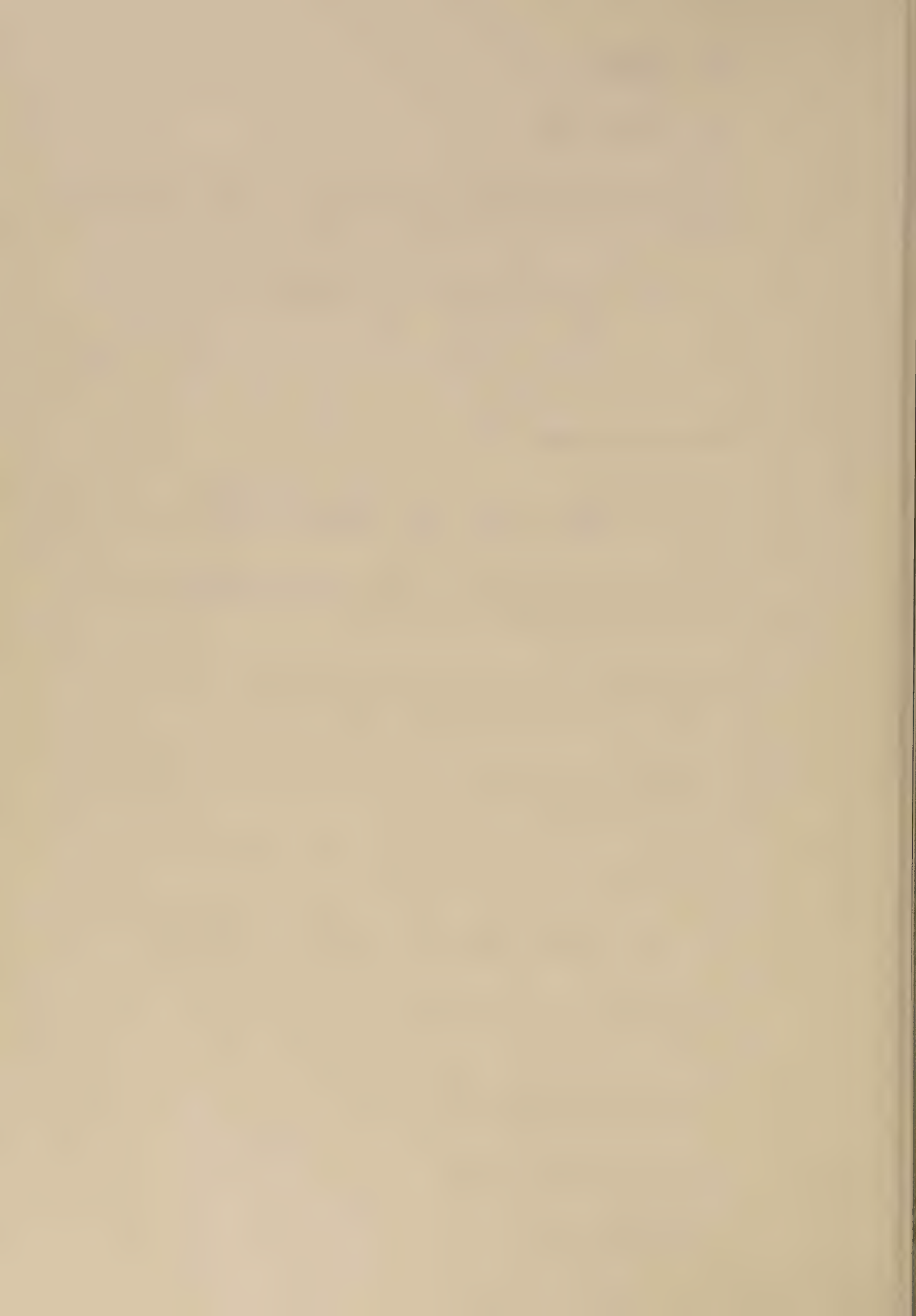
$$p = - \int_0^\xi \left[\frac{4}{\text{Re}} \frac{1}{r_0 e^\xi \sin \theta} \frac{\partial \Omega}{\partial \theta} + v_r \frac{\partial v_r}{\partial \xi} \right] d\xi + \begin{cases} p_{\xi=0, \theta=0} \\ p_{\xi=0, \theta=\pi} \end{cases}$$

2) On the surface of the sphere ($\xi = 0$)

$$p = \frac{4}{\text{Re}} \int_{\pi/2}^\theta \frac{1}{r_0 e^\xi \sin \theta} \frac{\partial \Omega}{\partial \xi} d\theta + p_{\text{ref}}$$

3) On the "outer" boundary

$$p = p_{\text{outer}}$$



4.2 "OUTER" FIELD

The pressure equation for the "outer" field is

$$\nabla^2 p - 4 \left[\frac{\partial v_z}{\partial z} \frac{\partial v_\rho}{\partial \rho} - \frac{\partial v_z}{\partial \rho} \frac{\partial v_\rho}{\partial z} - \left(\frac{v_\rho}{\rho} \right)^2 \right] = 0 \quad (20)$$

where ∇^2 is

$$\nabla^2 = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} \quad (21)$$

The velocity components are calculated from eqs (15) and (16).

The boundary conditions are

1) Along the axis of symmetry ($\rho = 0$)

$$p = - \int_{z_{\text{common}}}^z \left[\frac{4}{\text{Re}} \frac{1}{\rho} \frac{\partial \Omega}{\partial \rho} + v_z \frac{\partial v_z}{\partial z} \right] dz + p_{\text{common}}$$

where p_{common} is the pressure at $z = z_{\text{common}}$ in the "common" field.

2) Far away from the sphere ($z \rightarrow \pm\infty$)

$$p = p_{+\infty}$$

and

$$p = p_{-\infty}$$

3) On the cylinder wall ($\rho = 1$)

$$p = - \frac{4}{\text{Re}} \int_{-\infty}^z \frac{\partial \Omega}{\partial \rho} dz + p_{-\infty}$$

4) On the "inner" boundary

$$p = p_{\text{inner}}$$



5 DRAW ON THE SPHERE, PRESSURE DROP AND FORCE ON THE WALL

The drag on the body is expressed in terms of a dimensionless drag coefficient

$$C_D = F_D / \left(\frac{\pi d^2}{4} \cdot \frac{\rho U^2}{2} \right) \quad (22)$$

where F_D is the total drag on the sphere. The drag coefficient is composed of two parts, one due to friction drag and the other to pressure drag.

The friction drag coefficient, C_{Df} , is

$$C_{Df} = - \frac{1}{r_0} \frac{8}{Re} \int_0^\pi \Omega|_{\xi=0} \sin\theta \, d\theta \quad (23)$$

and the pressure drag coefficient, C_{Dp} , is

$$C_{Dp} = - 2 \int_0^\pi p|_{\xi=0} \cos\theta \sin\theta \, d\theta \quad (24)$$

The drag coefficient, C_D , then is

$$C_D = C_{Df} + C_{Dp} \quad (25)$$

The pressure drop, Δp , can be written

$$\Delta p = \Delta p^0 + \Delta p^+ \quad (26)$$

where Δp^0 is the pressure drop in the absence of the sphere and Δp^+ is the additional pressure drop caused by the presence of the sphere. The pressure drops are given by

$$\Delta p = p_{z=-L} - p_{z=L} \quad (27)$$

$$\Delta p^0 = p_{z=-L}^0 - p_{z=L}^0 \quad (28)$$



where the dimensionless L is a large positive number. The additional pressure drop, Δp^+ , for Poiseuille flow is then

$$\Delta p^+ = \lim_{L \rightarrow \infty} (p_{z=-L} - p_{z=L} - \frac{64}{Re} \cdot L) \quad (29)$$

The force on the wall can be expressed in a manner similar to that for the pressure drop

$$F_w = F_w^0 + F_w^+ \quad (30)$$

This force can be expressed in a dimensionless form by the equation

$$f = F_w / \left(\frac{\pi D^2}{4} \cdot \frac{\rho U^2}{2} \right) \quad (31)$$

and hence

$$f = f^0 + f^+ \quad (32)$$

The forces on the wall can be calculated from the expressions

$$f = \frac{8}{Re} \int_{-L}^L \Omega|_{\rho=1} dz \quad (33)$$

$$f^0 = \frac{8}{Re} \int_{-L}^L \Omega^0|_{\rho=1} dz \quad (34)$$

where L is a large positive number.

The additional force on the wall, f^+ , for Poiseuille flow is then calculated by

$$f^+ = \frac{8}{Re} \int_{-\infty}^{+\infty} (\Omega|_{\rho=1} - 4) dz \quad (35)$$

A momentum balance leads to a correlation between the drag coefficient, C_D , the additional pressure drop, Δp^+ , and the additional force on the wall, f^+ ,

$$\Delta p^+ = f^+ + r_0^2 \cdot C_D \quad (36)$$



6 CALCULATIONS

6.1 STREAMLINES AND ISOVORTICITY LINES

Calculations have been carried out for $Re = 0, 1, 5, 10, 25, 50, 100$ and 150 with the diameter ratio $d/D = 0.1$.

Equations (4), (5), (11) and (12) were expressed in finite difference form and solved simultaneously using a relaxation procedure. For the calculation of ψ overrelaxation was used and in the calculation of Ω account was taken of the Reynolds numbers, with overrelaxation for $Re < 50$ and underrelaxation for $Re \geq 50$. The results in this investigation have been calculated with a grid of 51×31 for the "inner" field; θ -increment, $h_\theta = \pi/30$, and ξ -increment, $h_\xi = 0.04$. For the "outer" field a grid of 151×26 was used for $Re \leq 10$, 301×26 for $25 \leq Re \leq 100$ and 351×26 for $Re = 150$. ρ - and z -increments were $h_\rho = h_z = 0.04$. Undisturbed Poiseuille flow was assumed 10 sphere diameters upstream of the sphere and 20, 50 or 60 sphere diameters downstream of the sphere. The method of solution consisted of applying the difference equations obtained from equations (4) and (5) to the "inner" field and the difference equations obtained from (11) and (12) to the "outer" field, and iteratively matching the two solutions for the "common" field. Convergence of the computed values of ψ and Ω was assumed when the change between consecutive iterations was less than 0.1%. ω was calculated from the accepted solution using equations (6) and (13).

6.2 ISOBARS

The pressure was calculated from the finite difference form of equations (18) and (20). The velocity components are related to the streamfunction according to equations (8), (9), (15) and (16). The equations were solved in a manner similar to the above and with the same convergence criterion. Overrelaxation was used for all Reynolds numbers.

The calculations have been carried out on the computer UNIVAC 1108 at Lund University Computing Center, Lund.

7 DISCUSSION

For $Re = 0$ (Stokes régime) there is upstream/downstream symmetry for streamlines, isovorticity lines and isobars. At higher Reynolds numbers the fore-and-aft asymmetry is most clearly illustrated by the isovorticity lines and isobars. The streamlines and isovorticity lines for $Re = 0, 1, 5, 10, 25, 50, 100$ and 150 are shown in figs 2 - 9 and the isobars in figs 10 - 17.

The calculations show that separation has not started at $Re = 100$. However, $\frac{\partial \omega}{\partial \theta} = 0$ at the point $\xi = 0, \theta = 0$ for $Re = 102$ (interpolated). This critical Reynolds number should be compared with the results for an unbounded flow around a sphere given by Le Clair et al. [7], who estimated the critical particle Reynolds number, Re_p , at 20, corresponding to $Re = 100$, and with those of Dennis and Walker [8], who obtained $Re_p = 20.5$ ($Re = 102.5$). The calculated length of the wake at $Re = 150$ is 0.031 (measured from the rear stagnation point) and the angle of separation is 26.2° ; see fig 18. The vorticity and pressure distributions on the surface of the sphere for different Reynolds numbers are given in figs 19 and 20.

The calculated drag coefficients, C_D, C_{Df}, C_{Dp} , the pressure difference over the sphere, $p_\pi - p_o$, and the additional pressure drop, Δp^+ , and the wall force, f^+ , caused by the presence of the sphere are presented in table 1 for various values of Reynolds numbers.

The wall effect is examined by comparing the calculated drag coefficient, C_D , and the pressure difference over the sphere, $p_\pi - p_o$, with results for an unbounded flow around a sphere (Le Clair et al. [7], Dennis and Walker [8]); see table 2 and fig 21. A wall correction factor is introduced, defined as $k_D = C_D/C_{Dunbounded}$ and $k_p = (p_\pi - p_o)/(p_\pi - p_o)_{unbounded}$ respectively. For $Re = 0$ the calculated wall correction factor for the drag, $k_D = 1.253$, is in good agreement with the analytically calculated value, $k_D = 1.255$, obtained from Bohlin's result [5] ($C_D = 602.2/Re$). The data shows that the wall effects become negligible for $Re > 50$.



Fidleris and Whitmore [9] have experimentally studied the wall effect for spheres falling axially in cylindrical vessels. They found that k_D became less than 1.01 for particle Reynolds numbers beyond 30 (corresponding to $Re = 150$) with the diameter ratio 0.1.

Fayon and Happel [10] have derived a semiempirical equation for the drag on a sphere in Poiseuille flow. For an axially situated, fixed sphere the equation is

$$C_D = \frac{1 - \frac{2}{3} r_o^2}{1 - 2.105 r_o + 2.087 r_o^3} C_S + (C_A - C_S) \quad (37)$$

where C_S is the Stokes drag and C_A the actual drag in an unbounded flow. Values of C_D calculated from eq (37) using values of C_A from Le Clair et al. and from Dennis and Walker are higher than those obtained in the investigation described here and show greater influence of the wall even at high Reynolds numbers.

The calculated additional pressure drop, Δp^+ , for $Re = 0$, $\Delta p^+ = 11.92/Re$, agrees with the analytically obtained result given by Happel and Byrne [3], $\Delta p^+ = 11.96/Re$. The momentum balance eq (36), is also very well satisfied.

Brenner [11] states that

$$\frac{\Delta p^+}{2 r_o^2 C_D} = 2 \left(1 - \frac{2}{3} r_o^2 \right) \quad (38)$$

for a sphere in a circular cylinder in Stokes régime. For $Re = 0$ the calculated value is 1.982 and eq (38) gives 1.987. Brenner also suggests that this relation should be valid for higher Reynolds number. The calculated values agree with the relation up to a value of Reynolds number about 25.



NOTATIONS

Symbols

C_A	drag coefficient, actual for an unbounded flow
C_D	drag coefficient
C_{Df}	friction drag coefficient
C_{Dp}	pressure drag coefficient
C_S	drag coefficient, from Stokes law
d	sphere diameter
D	tube diameter
E^2	operator, def. in eqs (7) and (14)
f	force on the wall, def. in eq (31)
F_D	drag force
F_w	force on the wall
h	grid spacing
k_D	wall correction factor for the drag
k_p	wall correction factor for the pressure
p	pressure [†] , dimensionless with respect to $\frac{1}{2} \rho U^2$
Δp	pressure drop, dimensionless with respect to $\frac{1}{2} \rho U^2$
r	spherical polar coordinate, dimensionless with respect to R
r_0	sphere radius, dimensionless with respect to R
R	tube radius
Re	tube Reynolds number ($Re = \frac{UD\rho}{\mu}$)
Re_p	particle Reynolds number ($Re_p = \frac{2Ud\rho}{\mu}$)
U	mean velocity in the undisturbed Poiseuille flow
v	velocity [†] , dimensionless with respect to U
z	circular cylindrical coordinate, dimensionless with respect to R
θ	spherical polar coordinate
μ	dynamic viscosity
ξ	spherical polar coordinate, def. as $\xi = \ln(r/r_0)$
ρ	circular cylindrical coordinate, dimensionless with respect to R or density
ψ	streamfunction, dimensionless with respect to UR^2
ω	vorticity, dimensionless with respect to U/R
Ω	def. in eqs (6) and (13)
∇	Nabla operator
∇^2	Laplace operator

[†] not dimensionless in eqs (1) - (3)



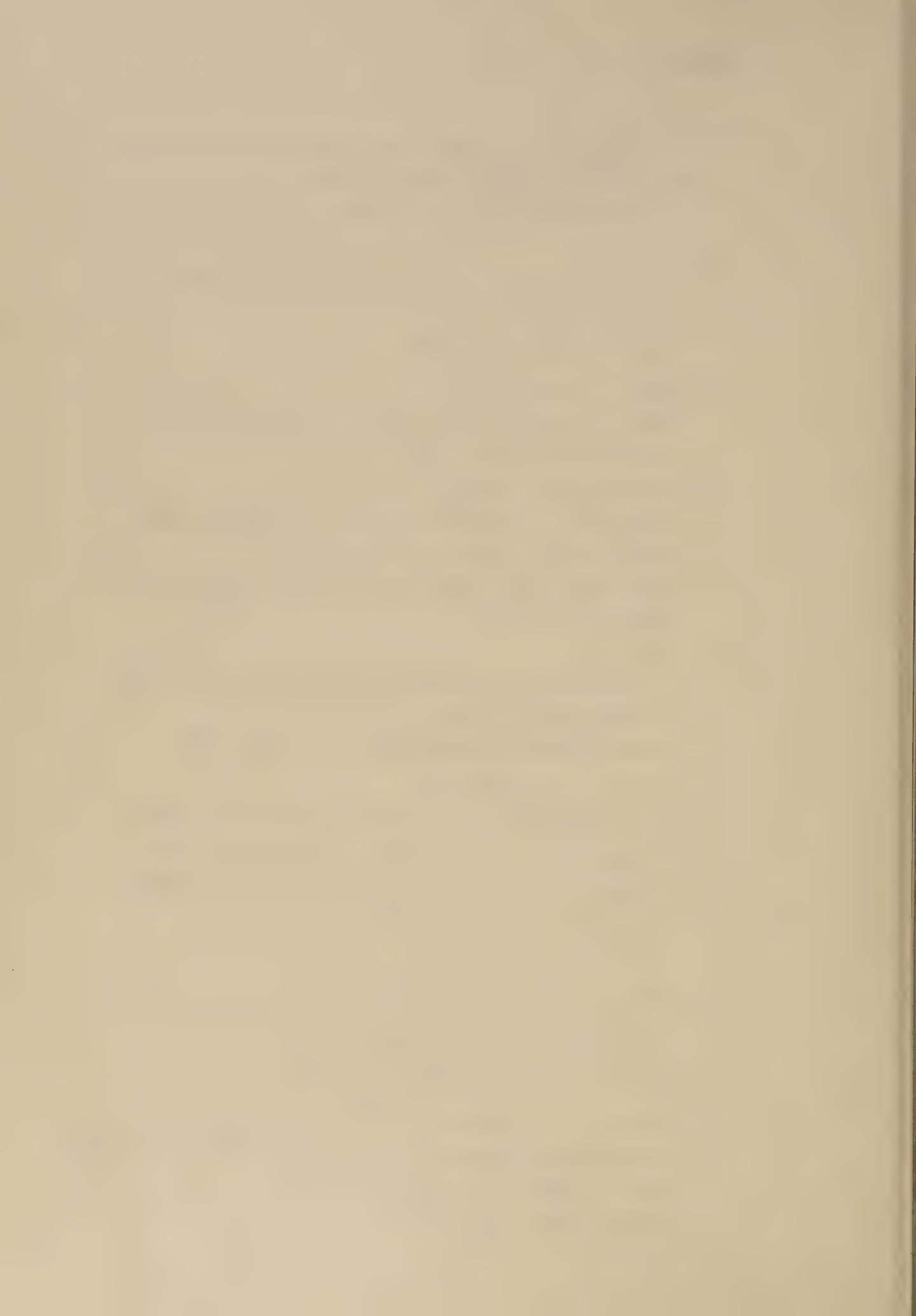
Superscripts

- vector
- + additional, in presence of the sphere
- o in absence of the sphere



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Re	C_D	C_{Df}	C_{Dp}	C_{Df}/C_{Dp}	$p_{\pi}^{-p_O}$	f^+	Δp^+
0	601.4/Re	396.9/Re	204.5/Re	1.941	312.7/Re	5.902/Re	11.92/Re
1	601.4	396.9	204.5	1.941	312.7	5.911	11.92
5	122.6	80.85	41.73	1.938	63.83	1.209	2.435
10	63.28	41.68	21.60	1.930	33.22	0.6191	1.252
25	28.91	18.91	10.00	1.891	15.69	0.2835	0.5726
50	17.20	11.10	6.105	1.818	10.01	0.1634	0.3354
100	10.84	6.797	4.038	1.683	7.233	0.0958	0.2042
150	8.430	5.170	3.260	1.586	6.296	0.0695	0.1538

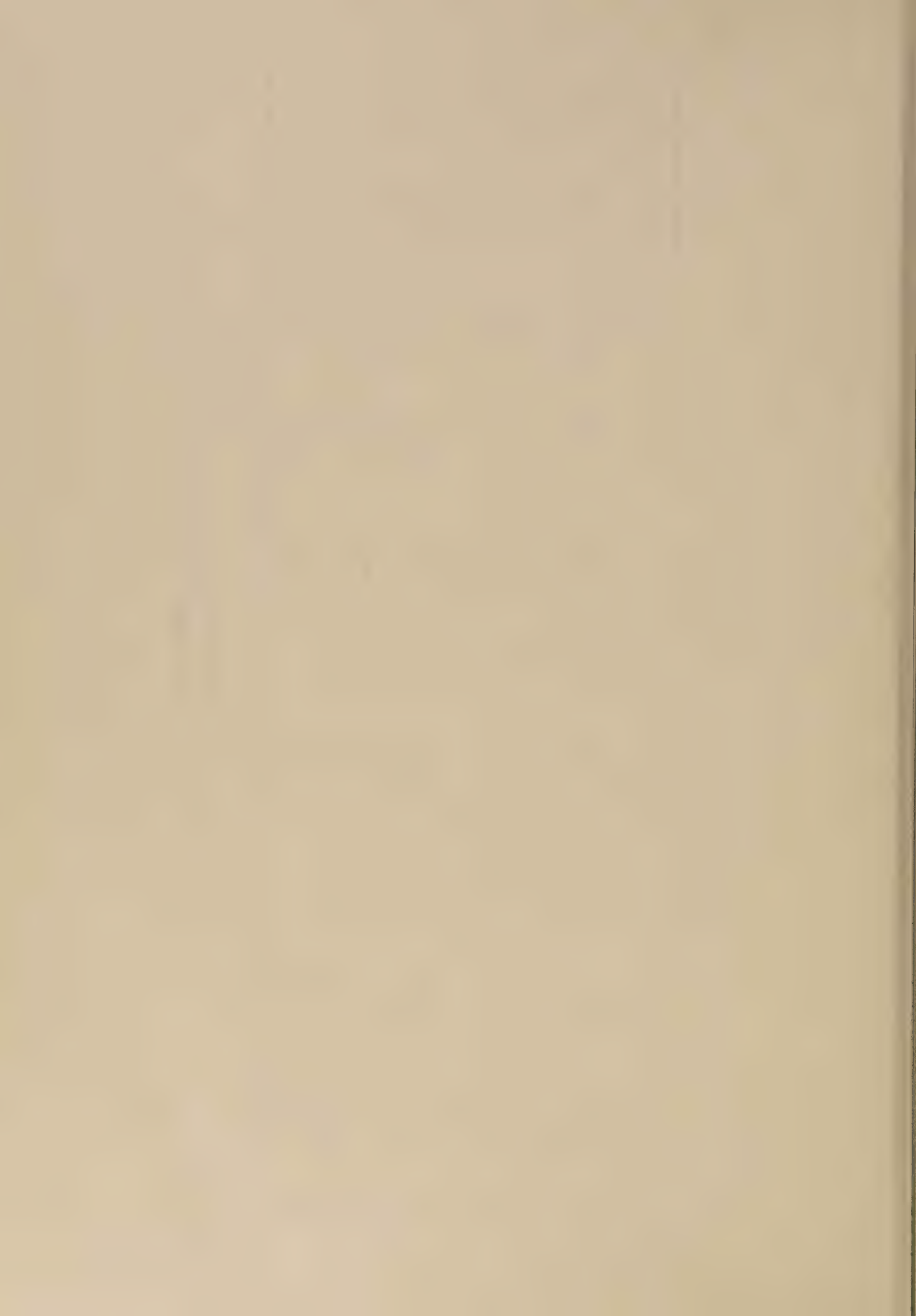
TABLE 1. Calculated properties.



Re	C_D	C_D^1	C_D^2	k_D^1	k_D^2	$p_\pi - p_o$	$p_\pi - p_o^1$	$p_\pi - p_o^2$	k_p^1	k_p^2
0	601.4/Re	480.0/Re	480.0/Re	1.253	1.253	312.7/Re	240.0/Re	240.0/Re	1.303	1.303
1	601.4	-	496.2	-	1.212	312.7	-	248.1	-	1.261
5	122.6	109.3	109.8	1.122	1.117	63.83	54.61	55.08	1.169	1.159
10	63.28	-	-	-	-	33.22	-	-	-	-
25	28.91	28.12	28.84	1.028	1.002	15.69	15.06	15.21	1.042	1.032
50	17.20	17.15	17.70	1.003	0.972	10.01	9.900	10.13	1.011	0.988
100	10.84	10.84	10.92	1.000	0.992	7.233	7.180	7.172	1.007	1.009
150	8.430	8.440	-	0.999	-	6.296	6.252	-	1.007	-

TABLE 2. Comparison of calculated C_D and $p_\pi - p_o$ with results obtained for an unbounded flow around a sphere by

¹ Le Clair et al. [7]
² Dennis and Walker [8]



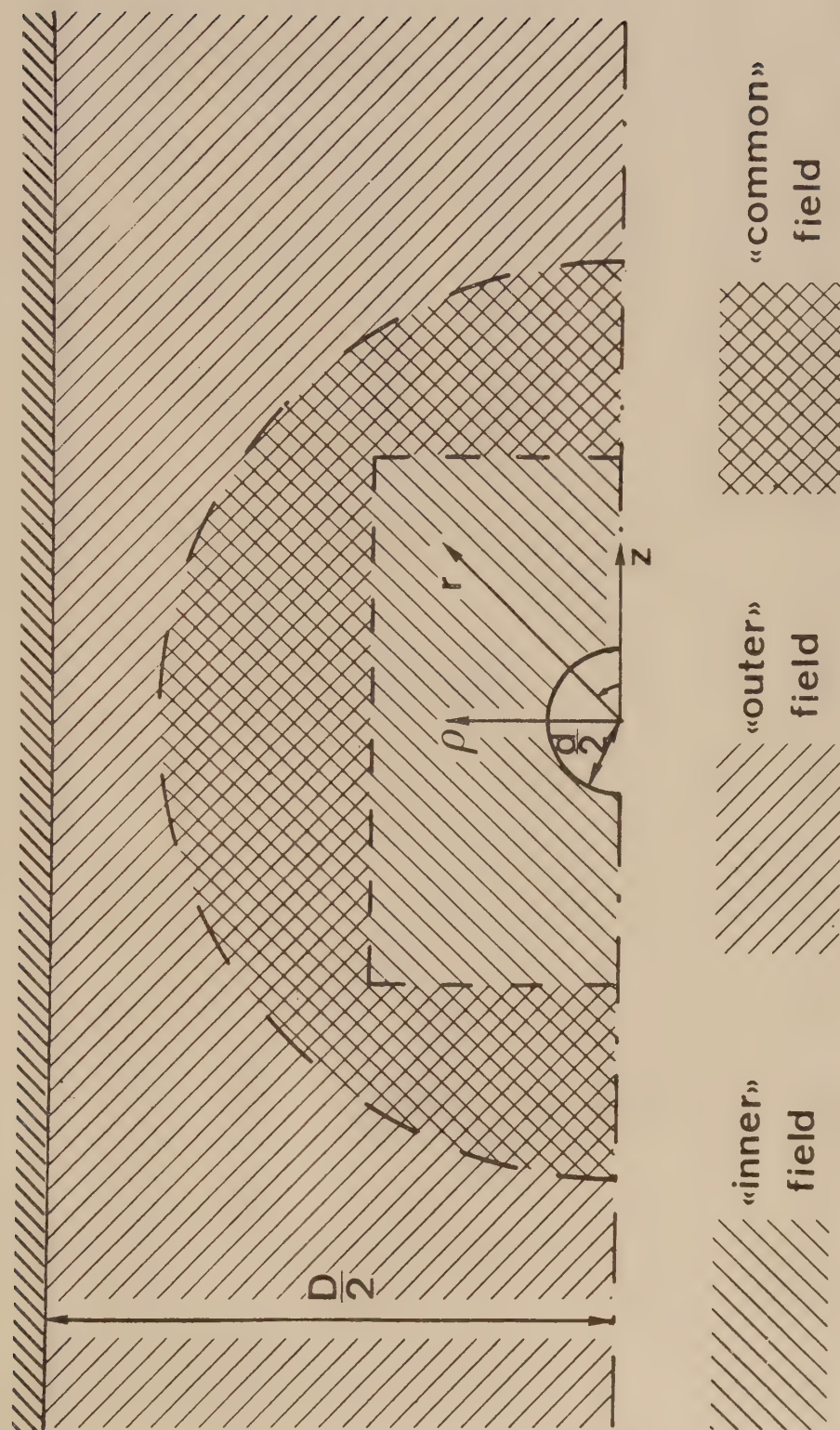


FIGURE 1. Computational fields.

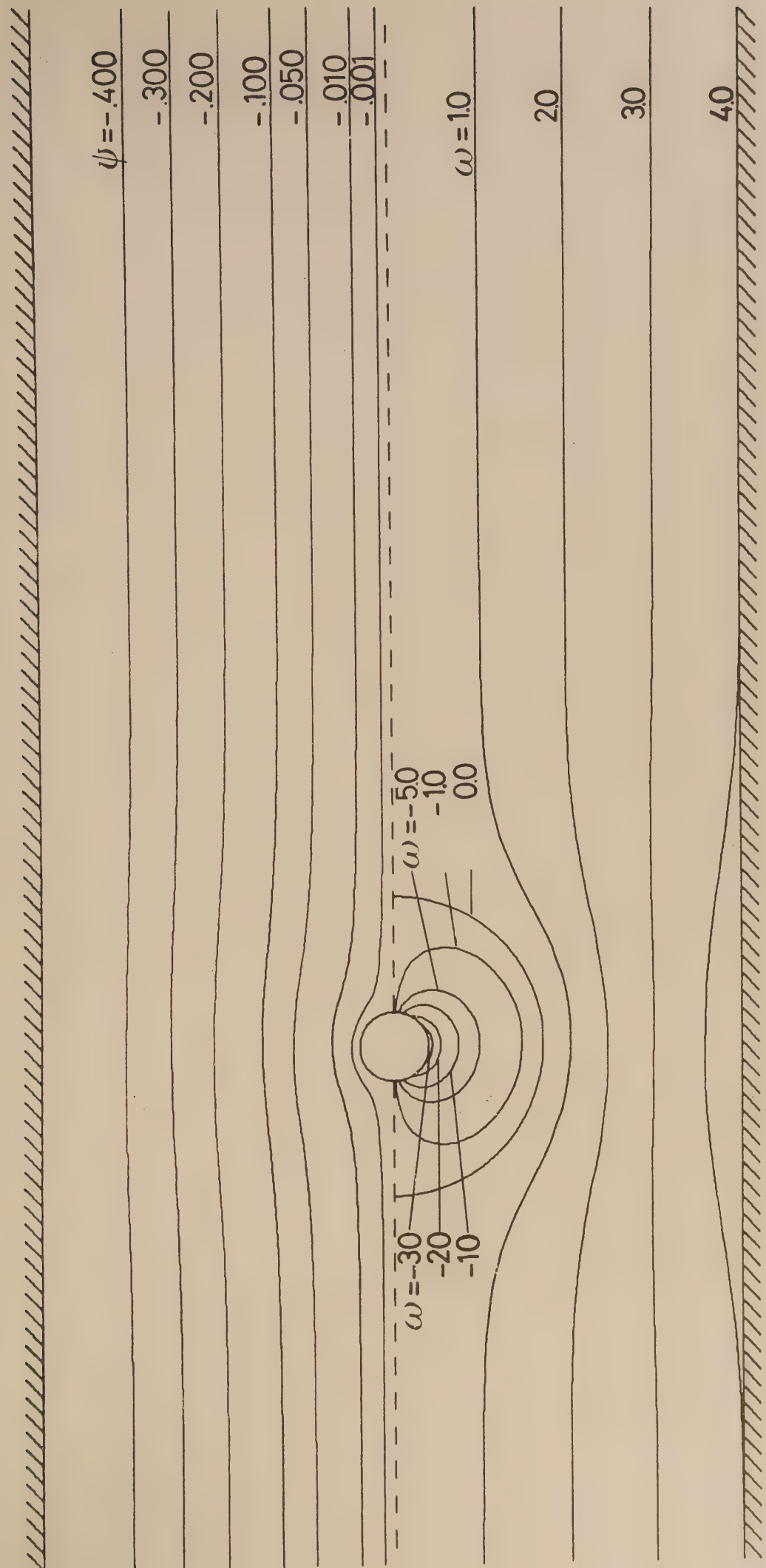


FIGURE 2. Streamlines and isovorticity lines for $Re = 0$.



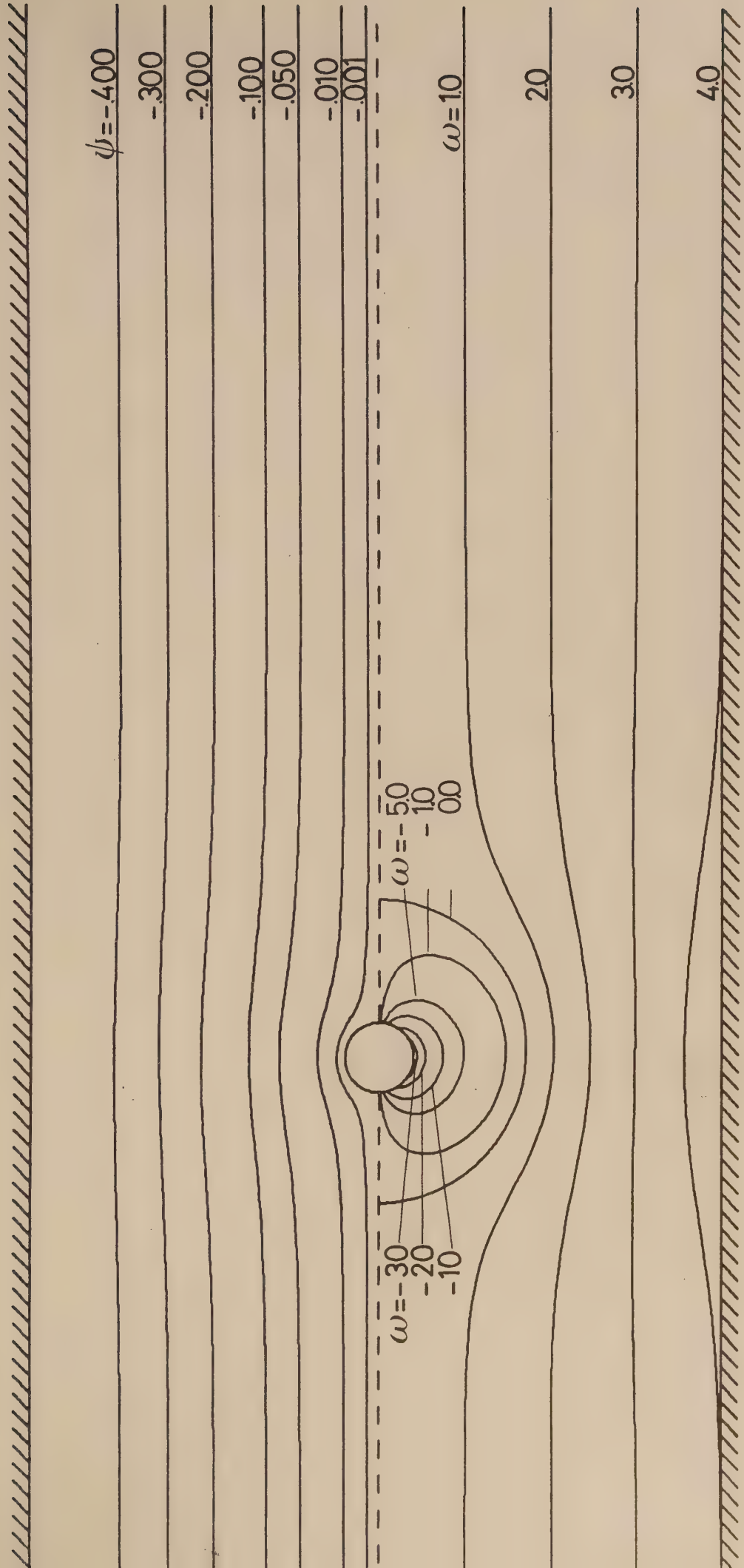


FIGURE 3. Streamlines and isovorticity lines for $Re = 1$.



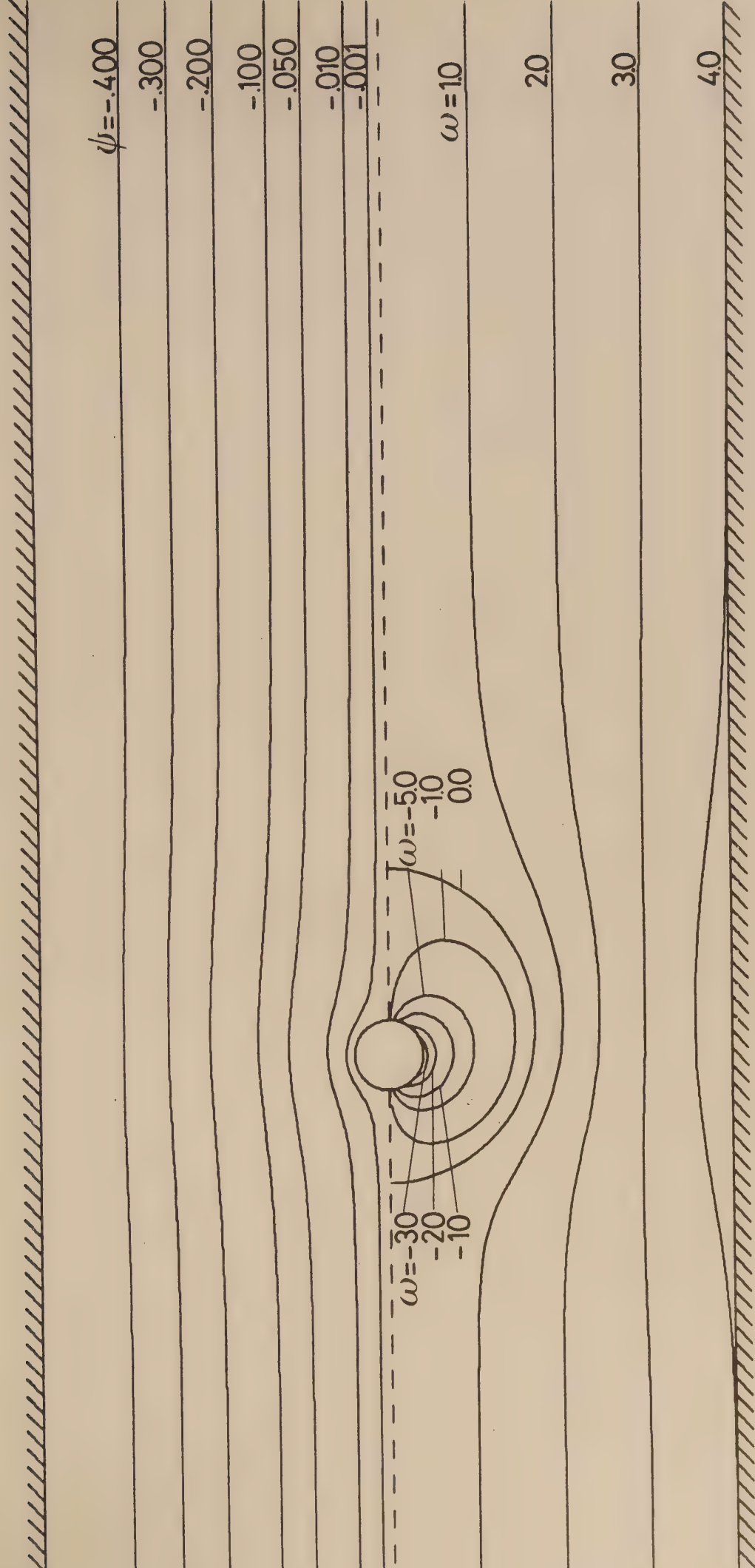


FIGURE 4. Streamlines and isovorticity lines for $Re = 5$.



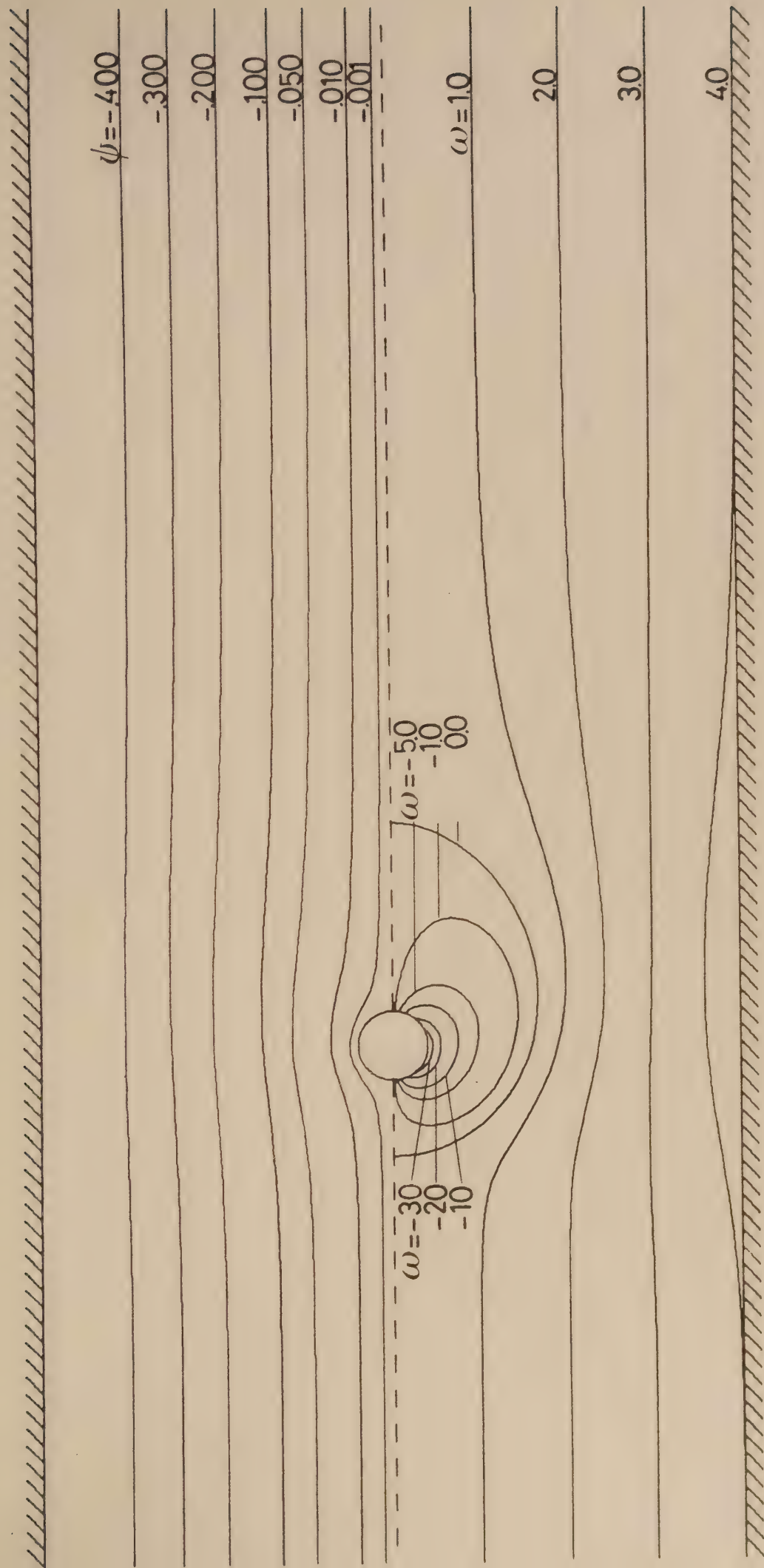


FIGURE 5. Streamlines and iso-vorticity lines for $Re = 10$.



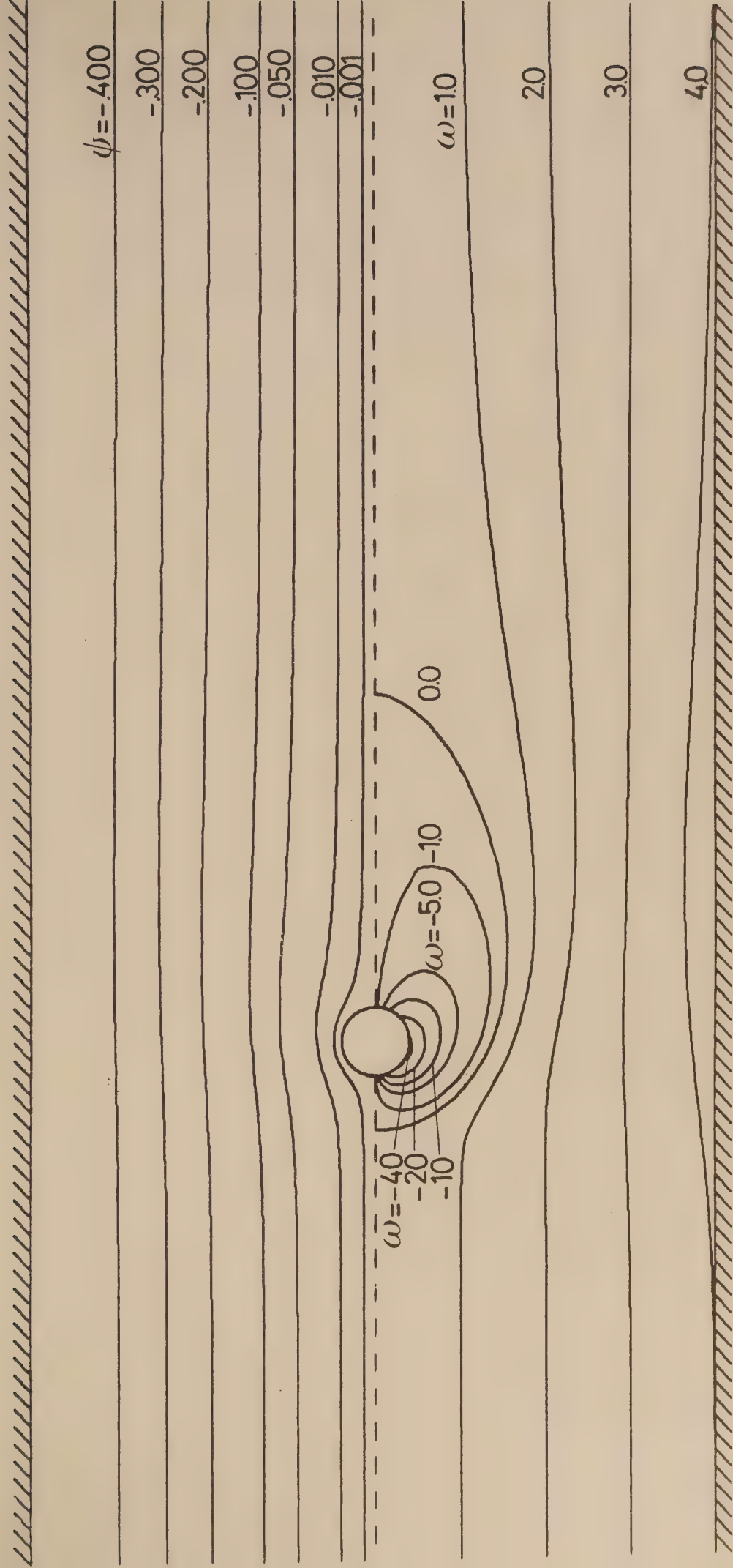


FIGURE 6. Streamlines and isovorticity lines for $Re \approx 25$.



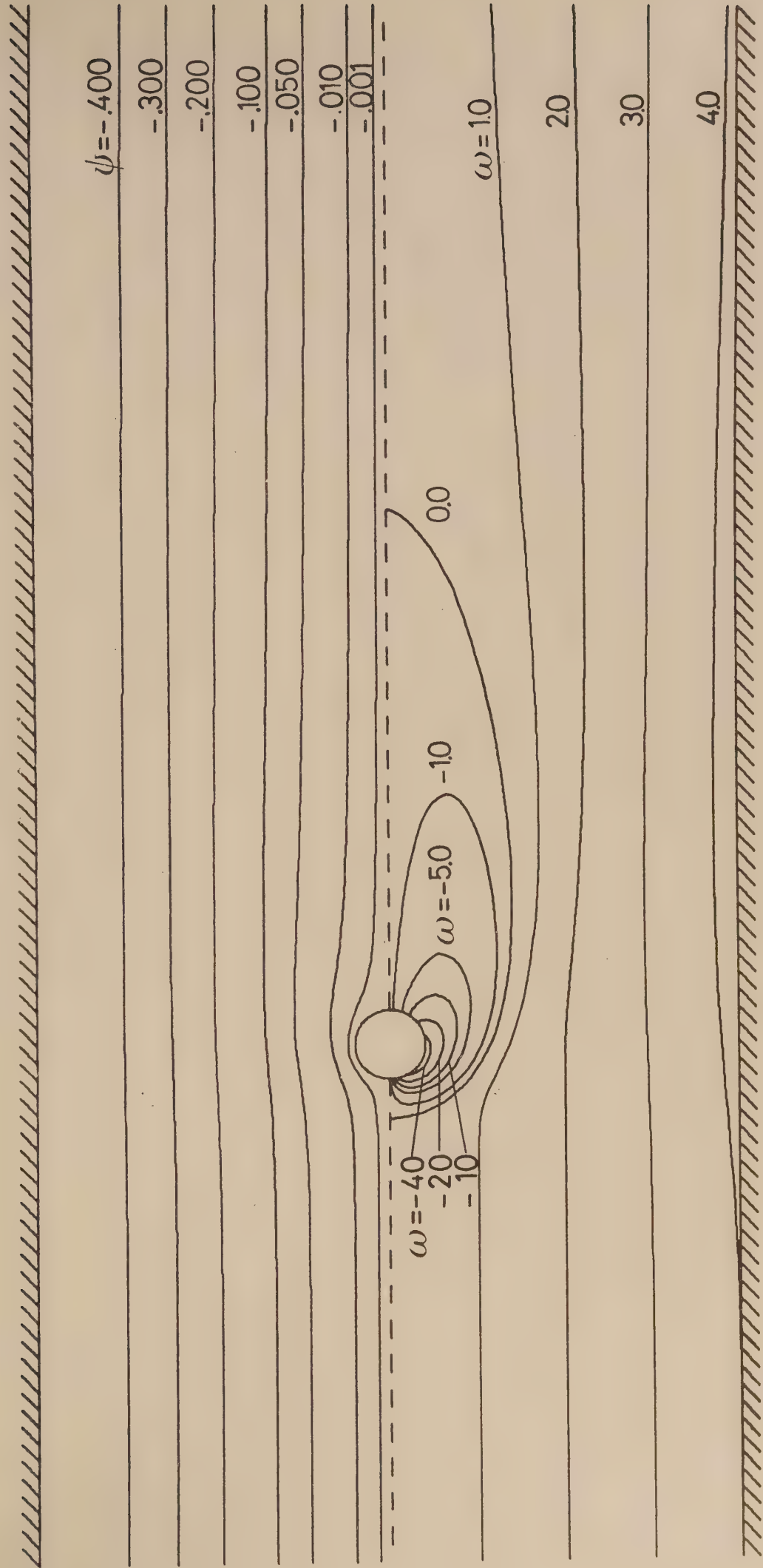


FIGURE 7. Streamlines and isovorticity lines for $Re = 50$.



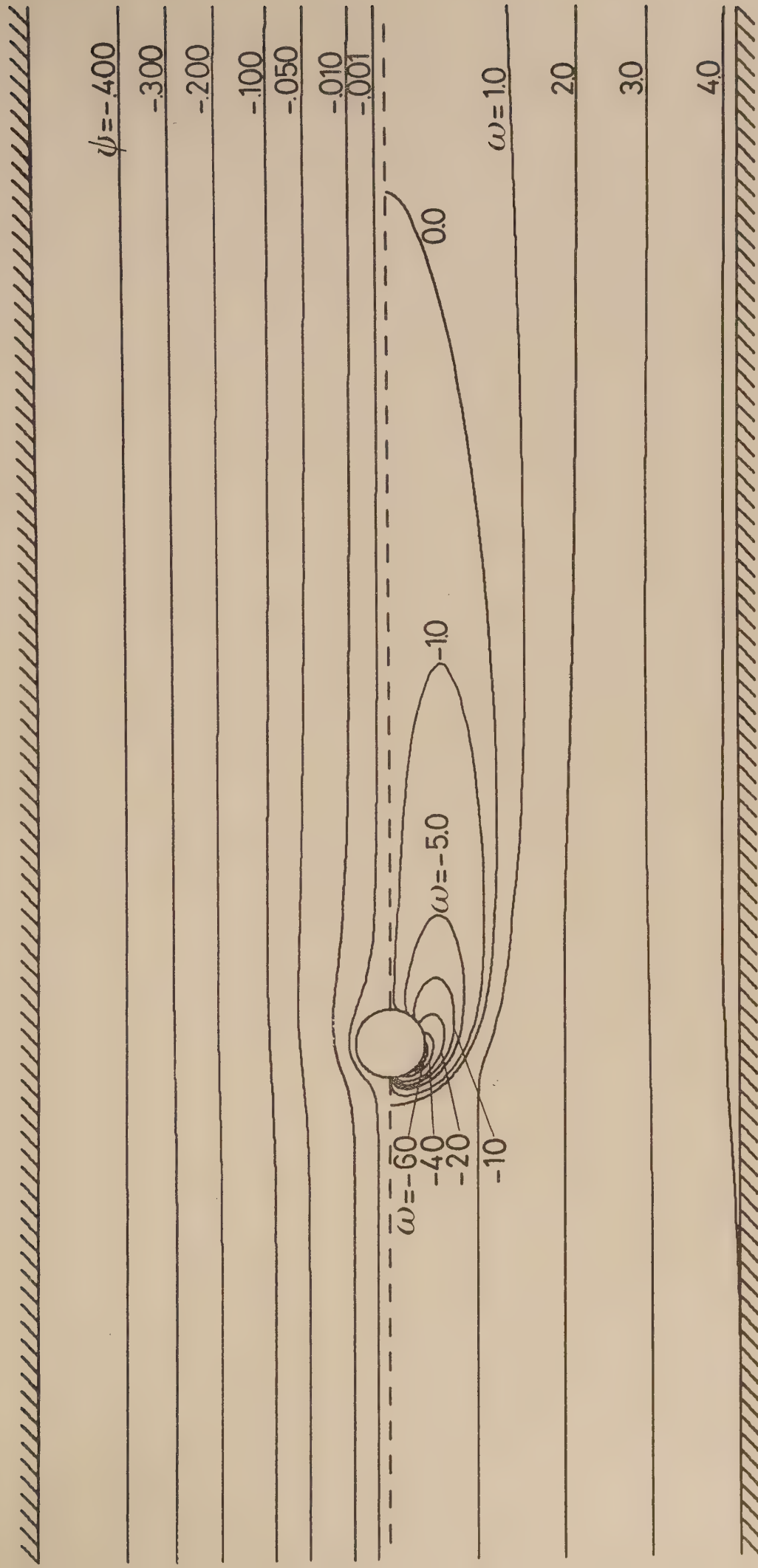


FIGURE 8. Streamlines and iso-vorticity lines for $Re = 100$.



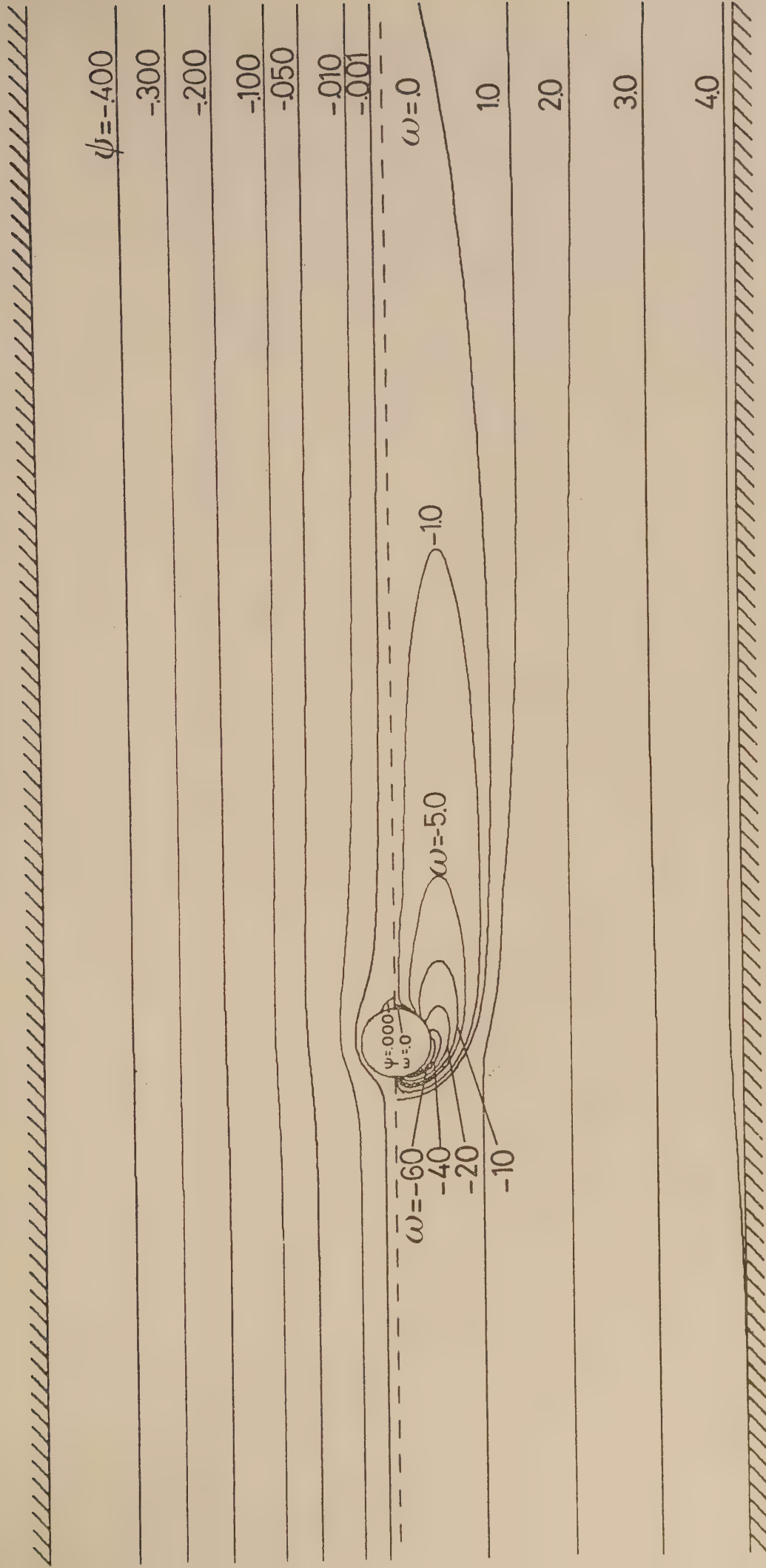


FIGURE 9. Streamlines and isovorticity lines for $Re = 150$.



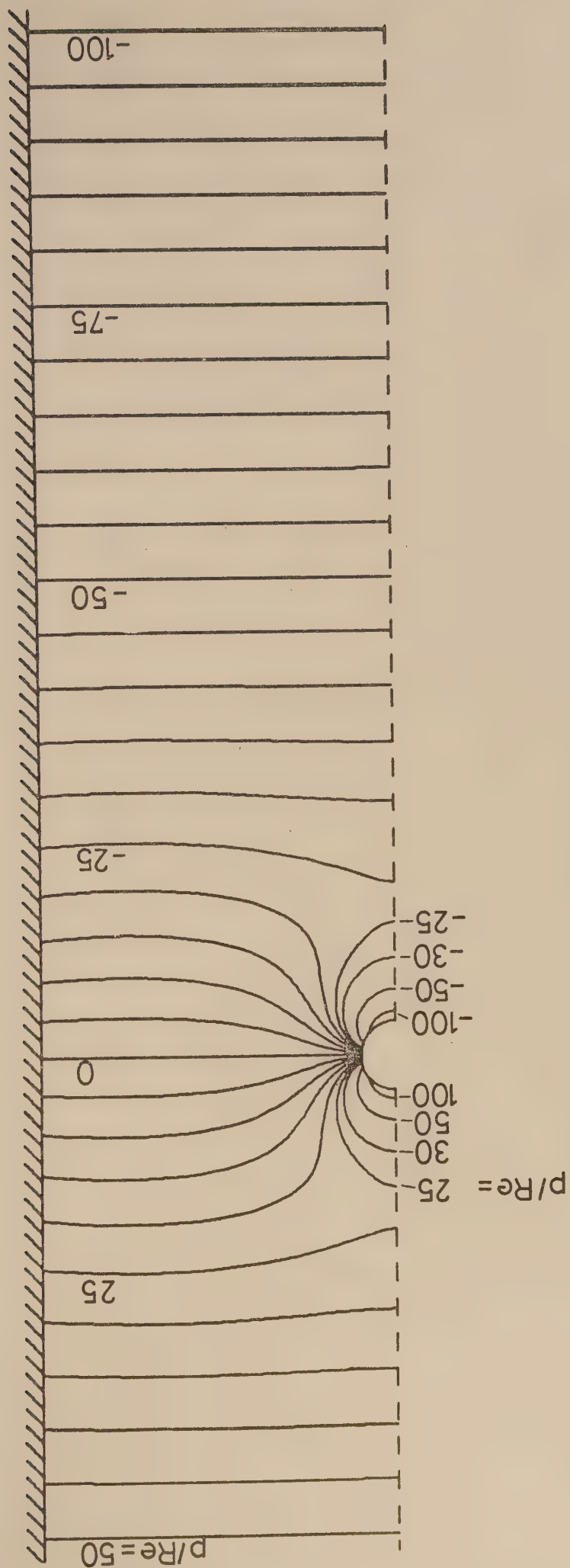


FIGURE 10. Isobars for $Re = 0$.



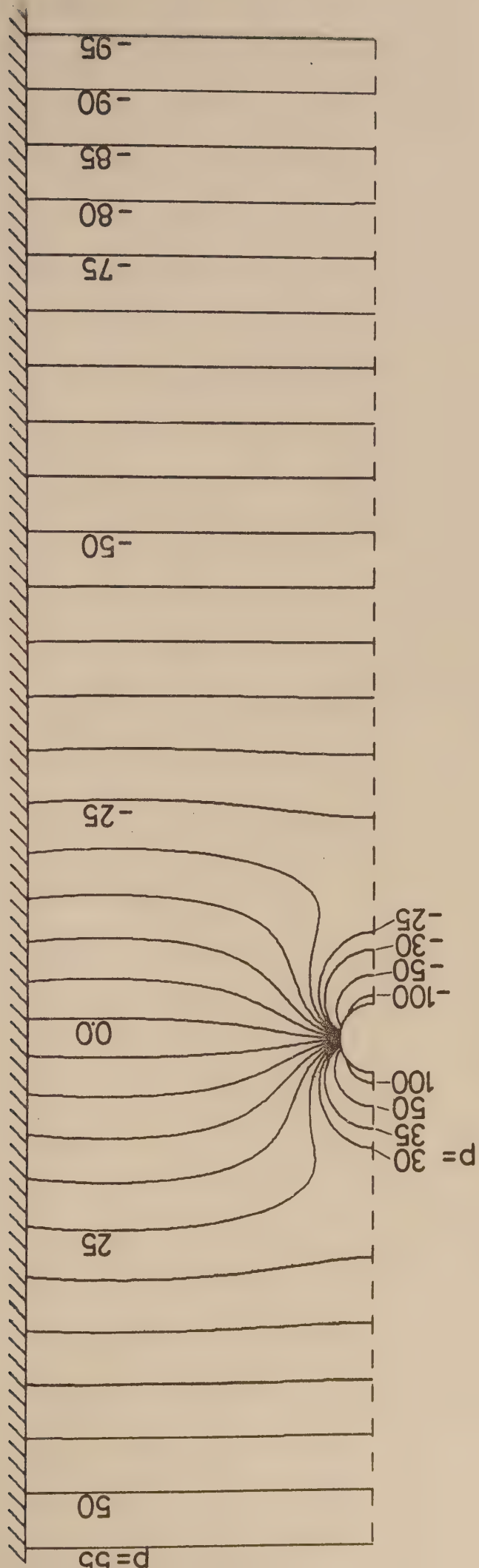


FIGURE 11. Isobars for $Re = 1$.



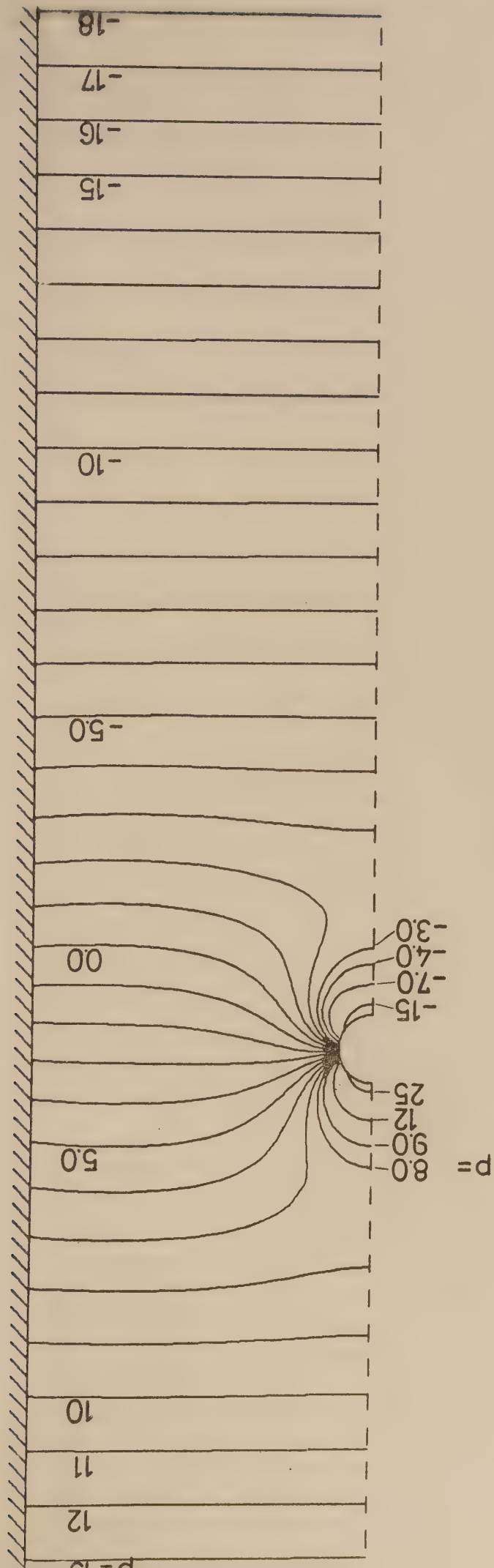


FIGURE 12. Isobars for $Re = 5$.



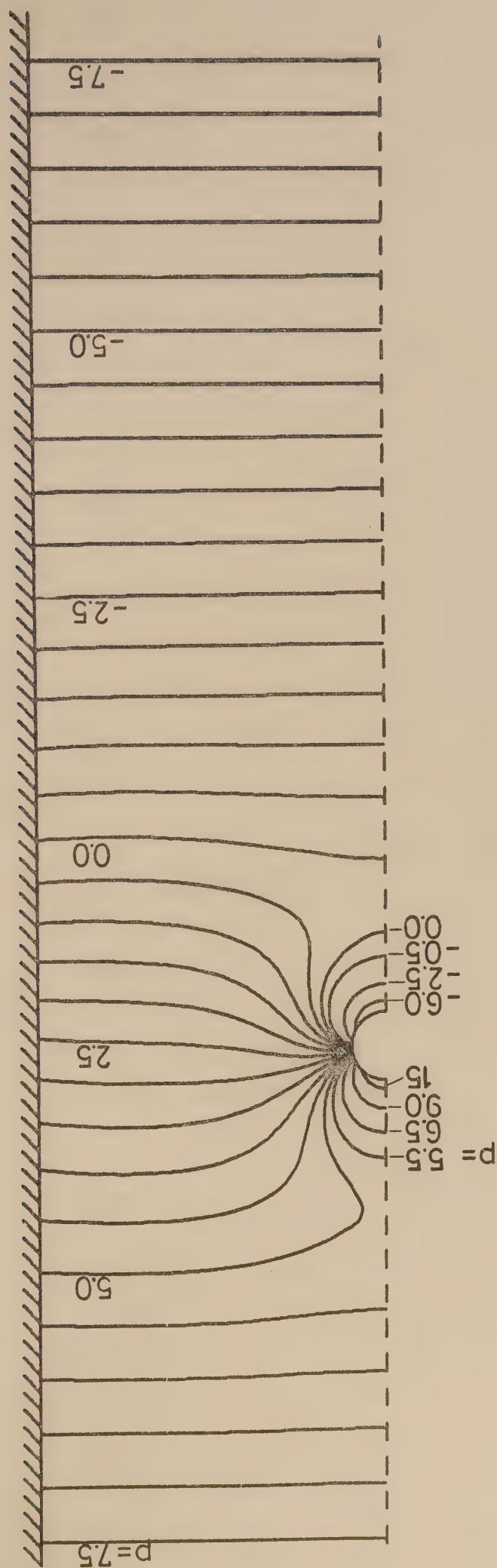


FIGURE 13. Isobars for $Re = 10$.



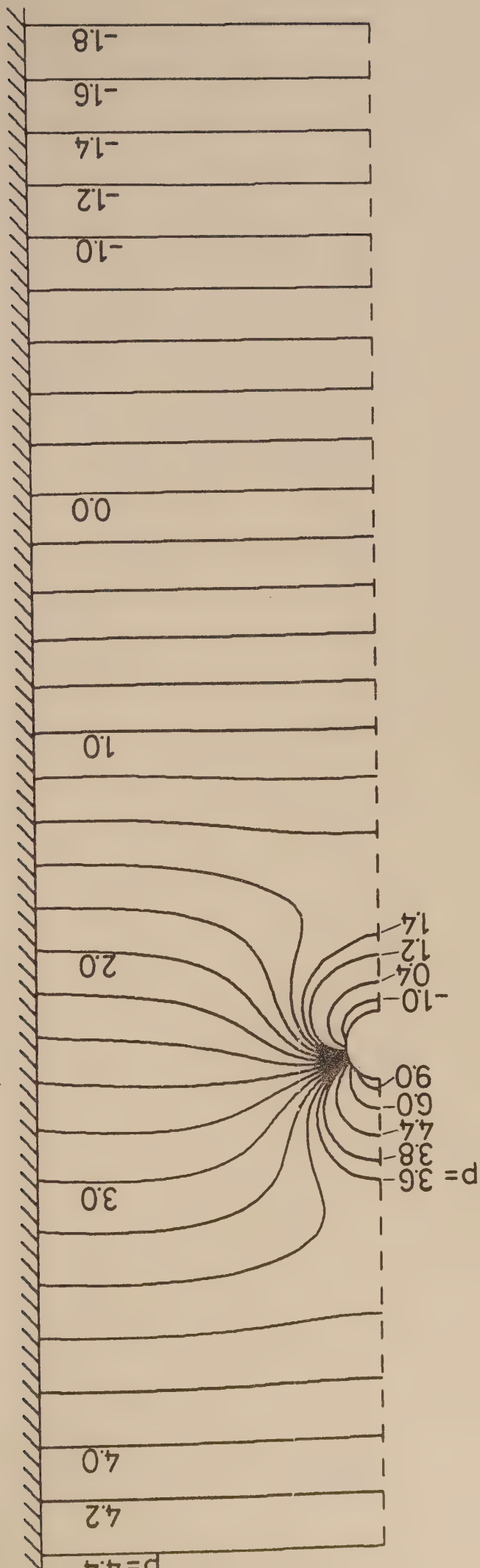
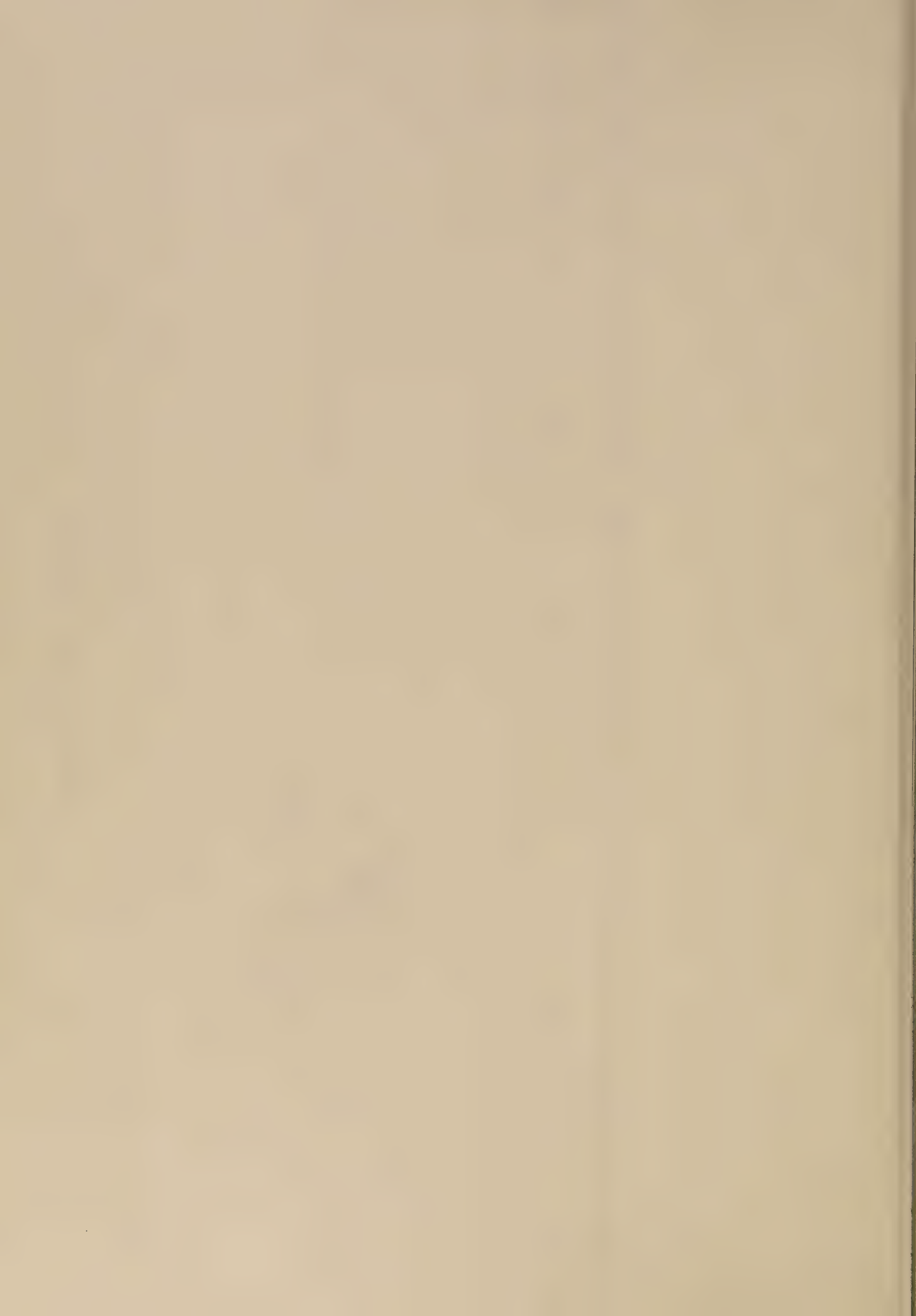


FIGURE 14. Isobars for $Re = 25$.



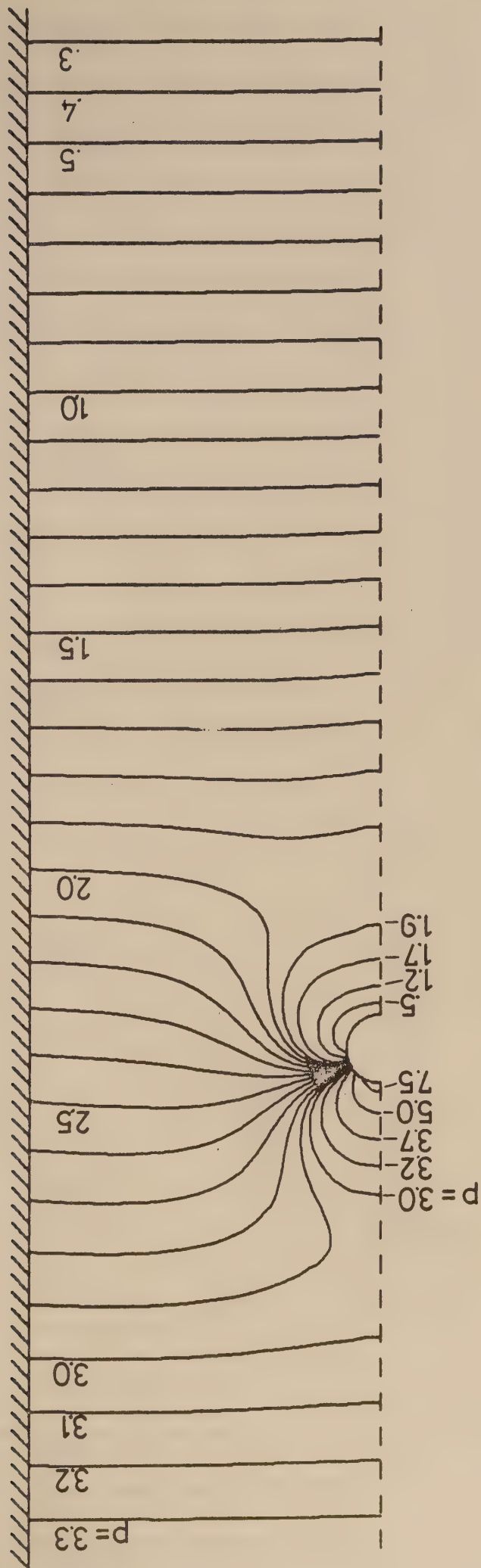


FIGURE 15. Isobars for $Re = 50$.



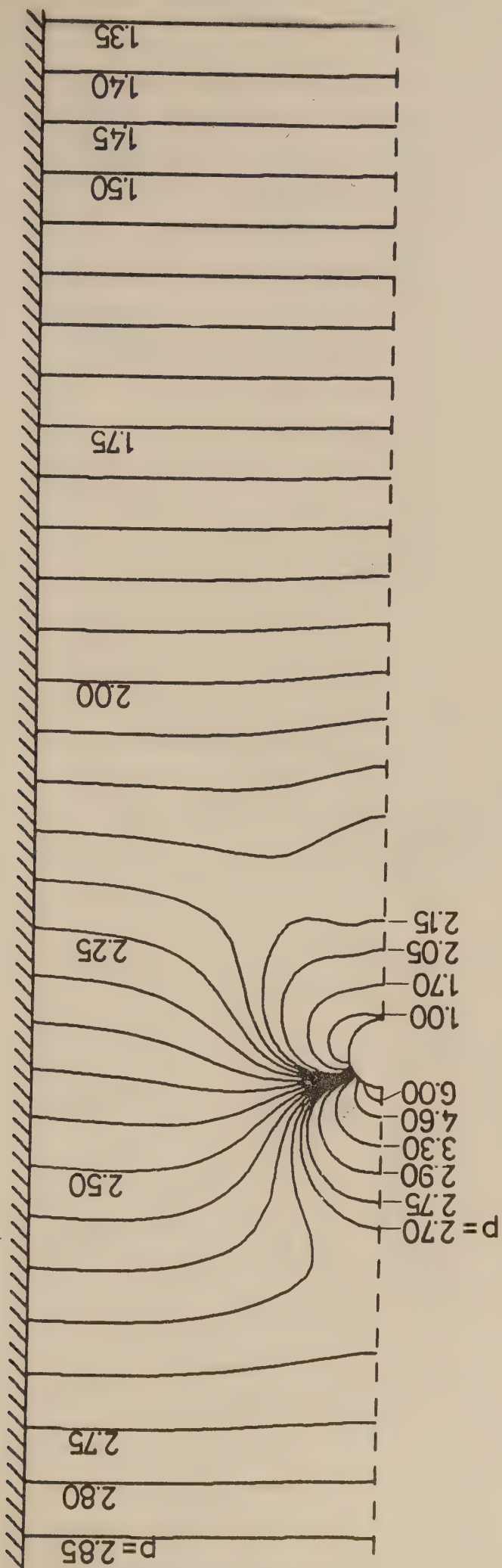


FIGURE 16. Isobars for $Re = 100$.



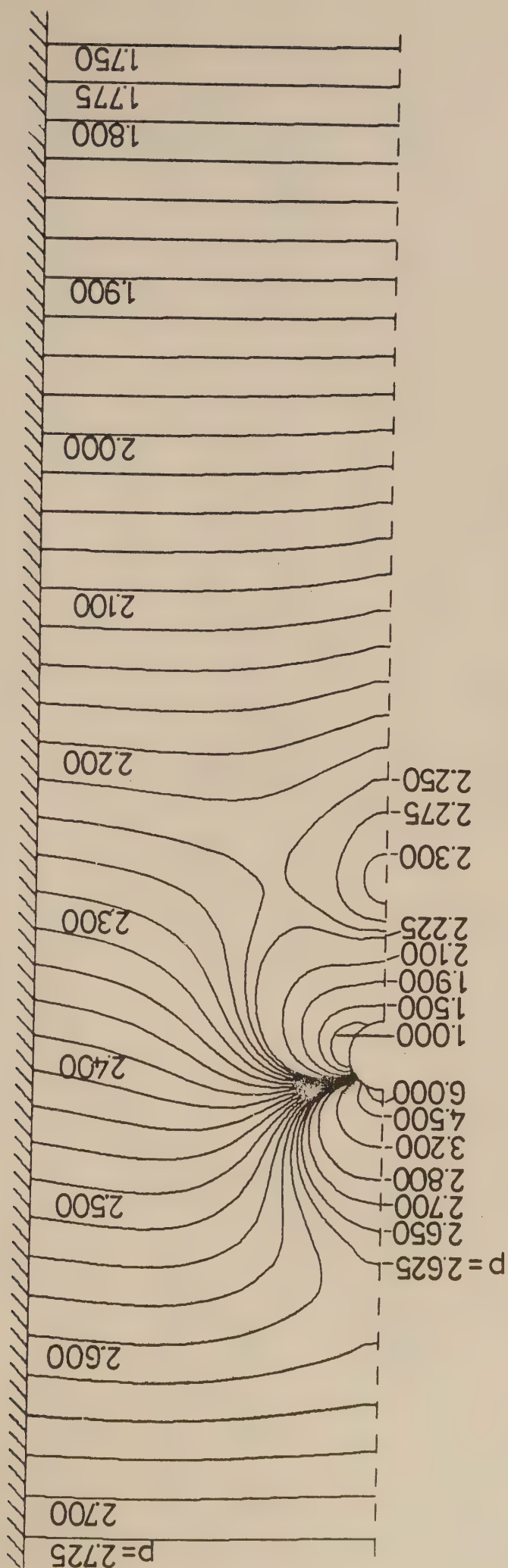
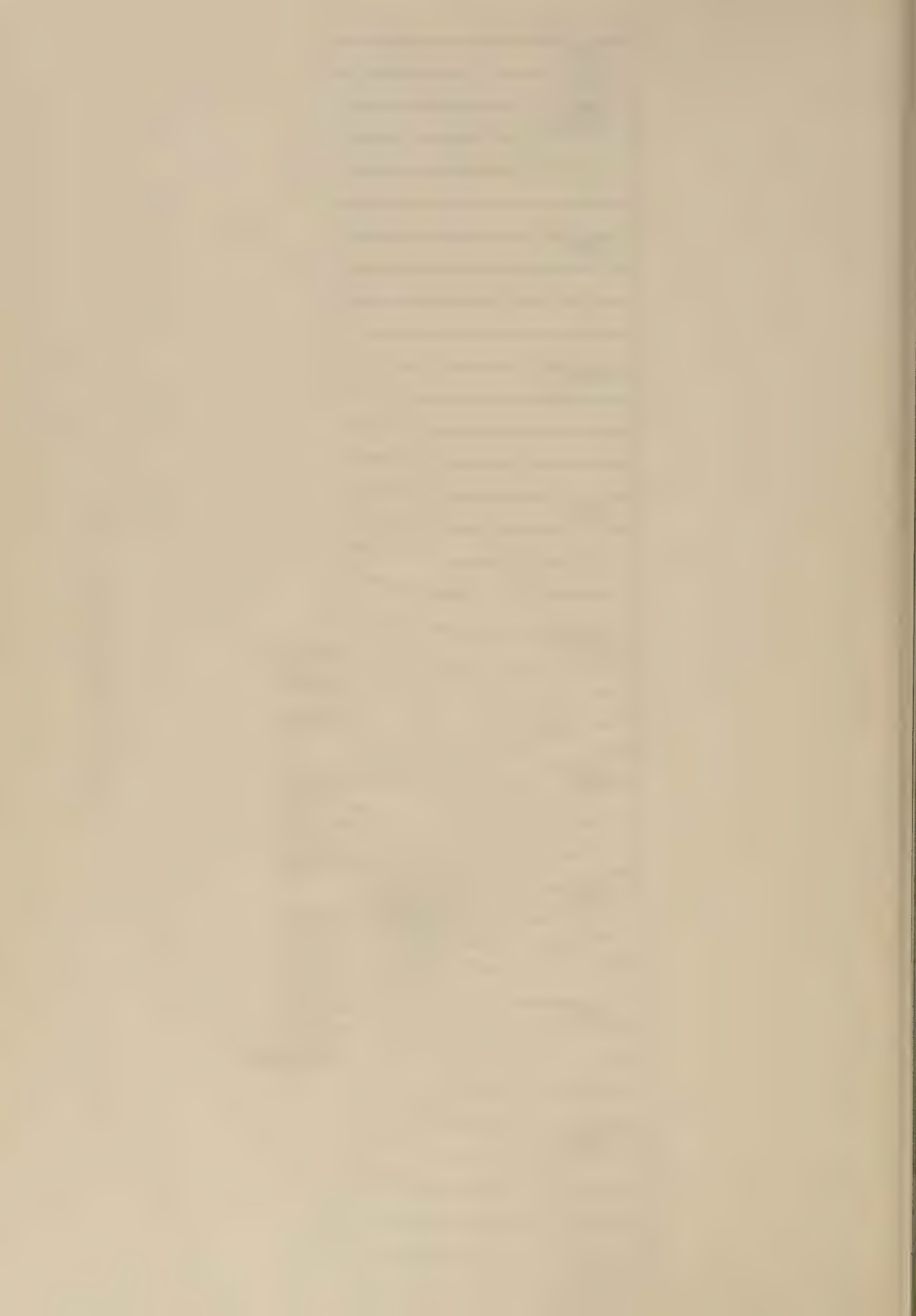


FIGURE 17. Isobars for $Re = 150$.



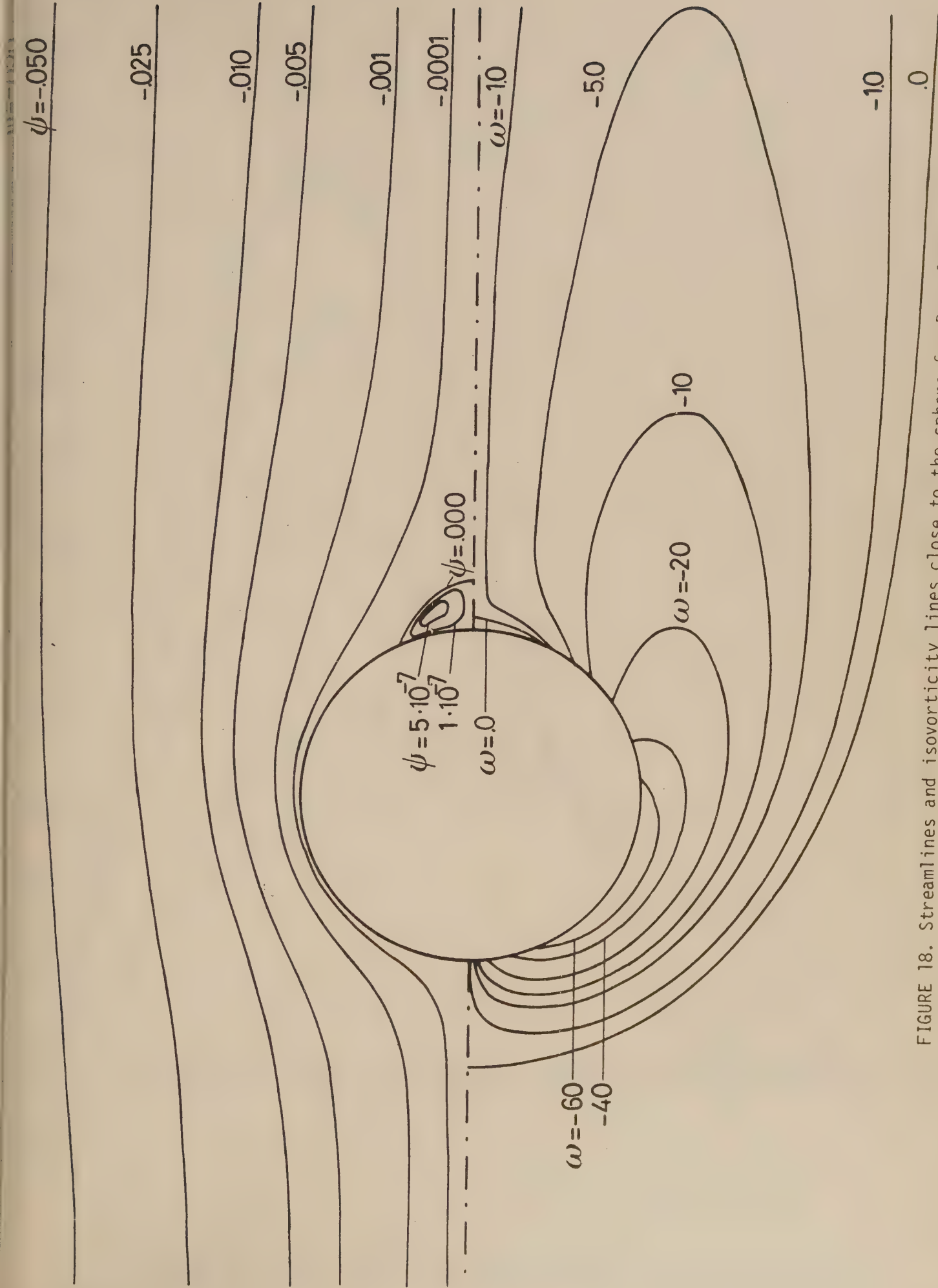


FIGURE 18. Streamlines and isovorticity lines close to the sphere for $Re = 150$.



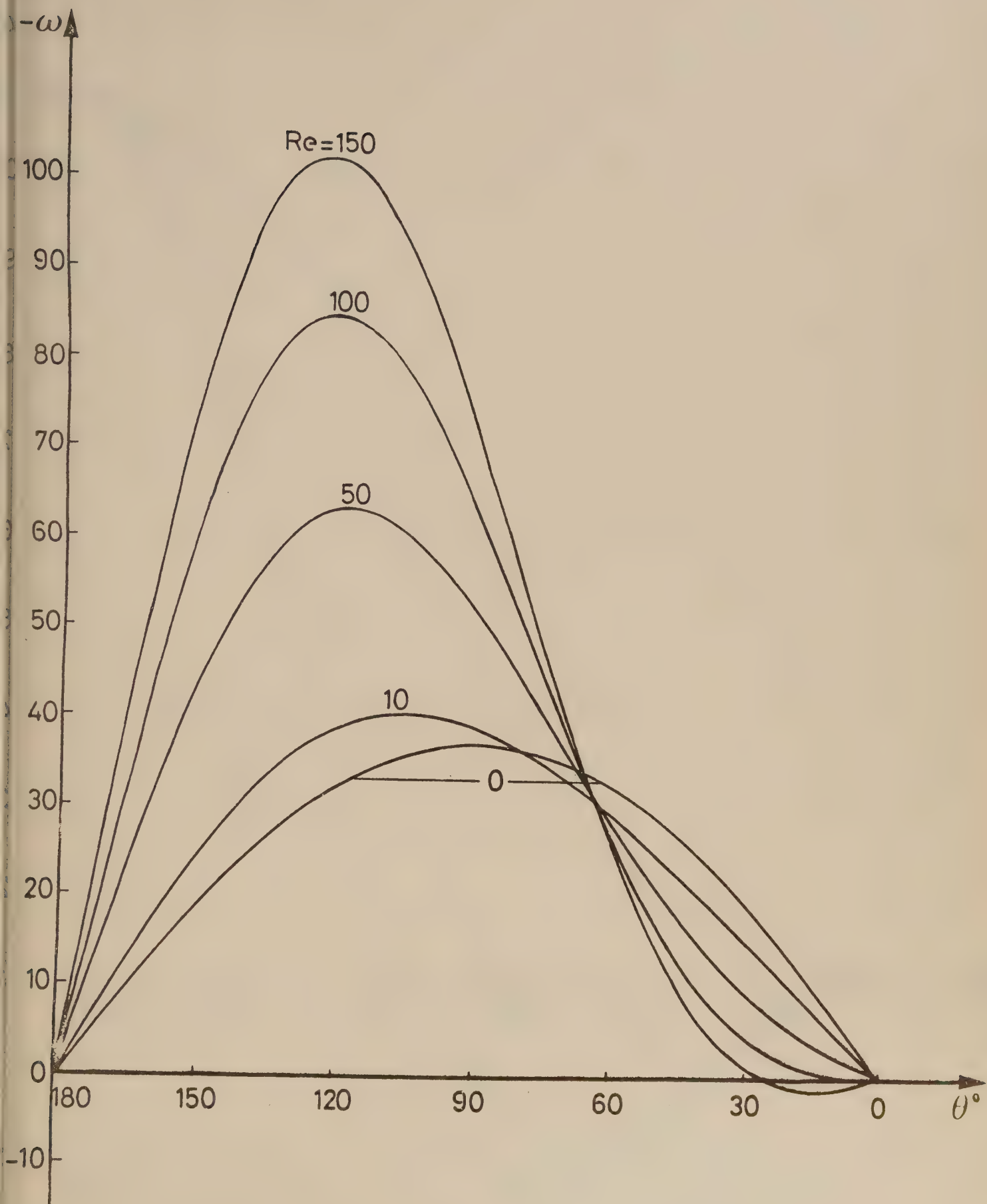


FIGURE 19. The vorticity distribution on the surface of the sphere.



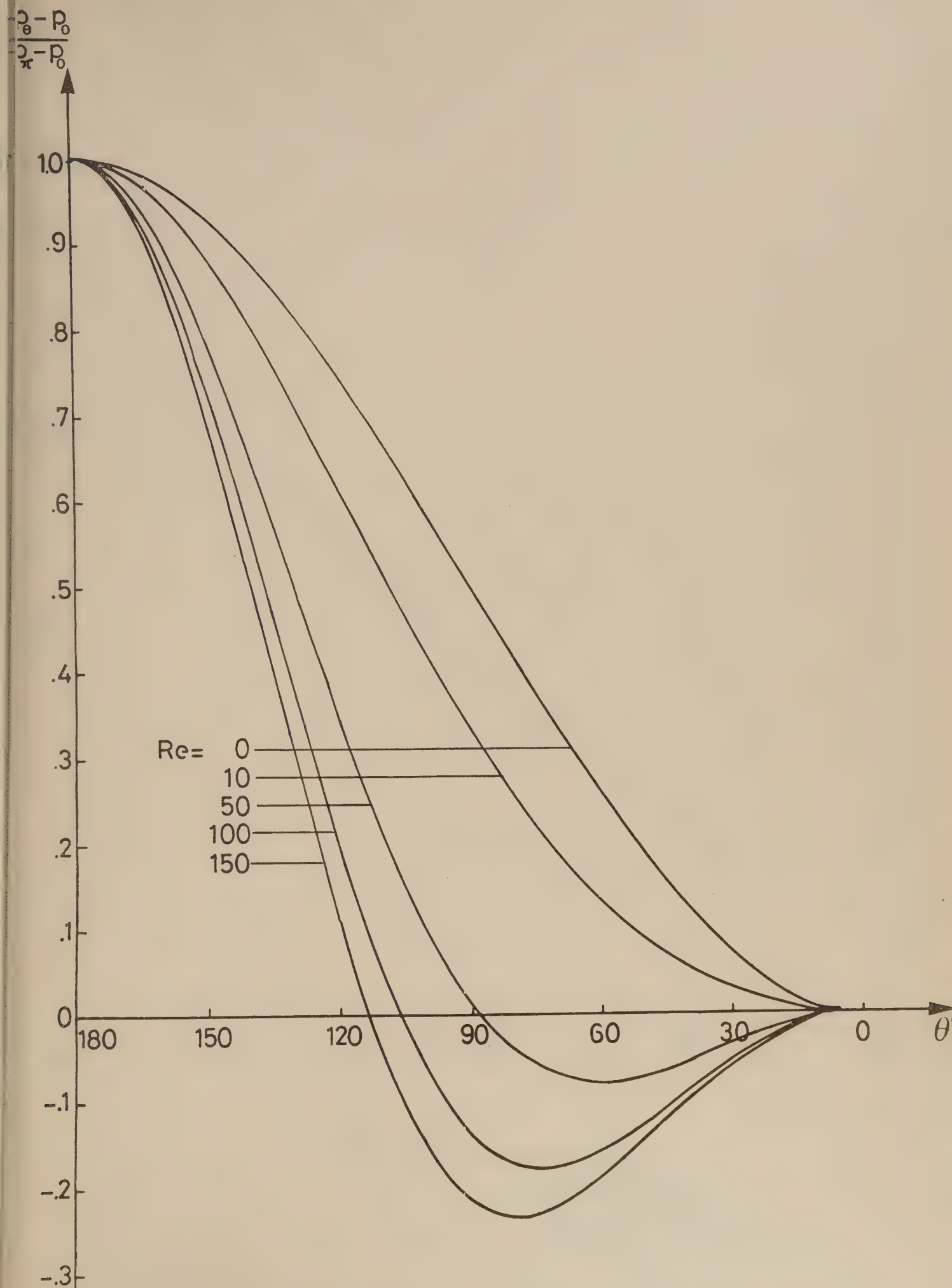


FIGURE 20. The pressure distribution on the surface of the sphere.



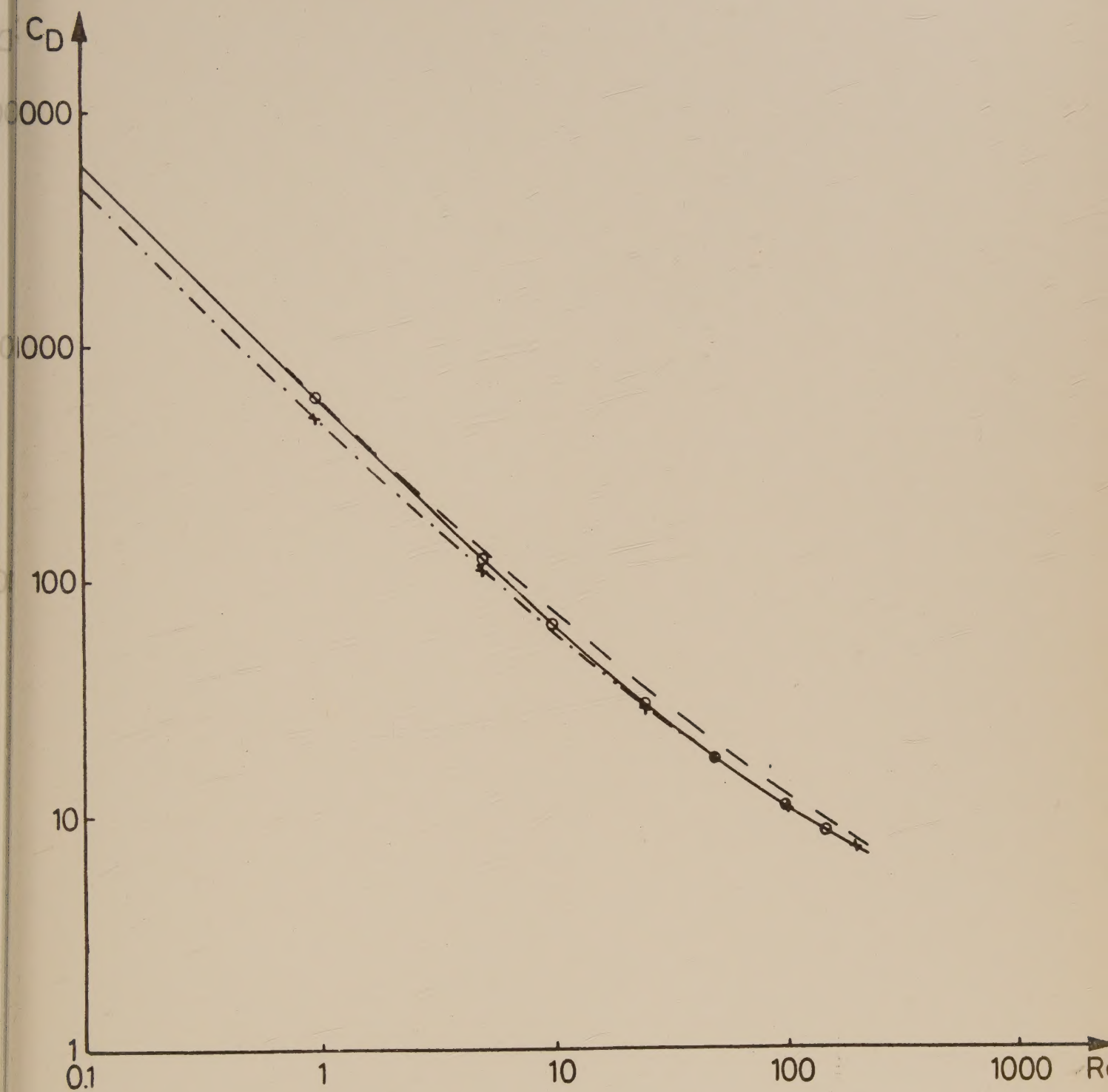


FIGURE 21. The drag coefficient.

- Fayon and Happel [10]
- . - . . Dennis and Walker [7]
- This investigation

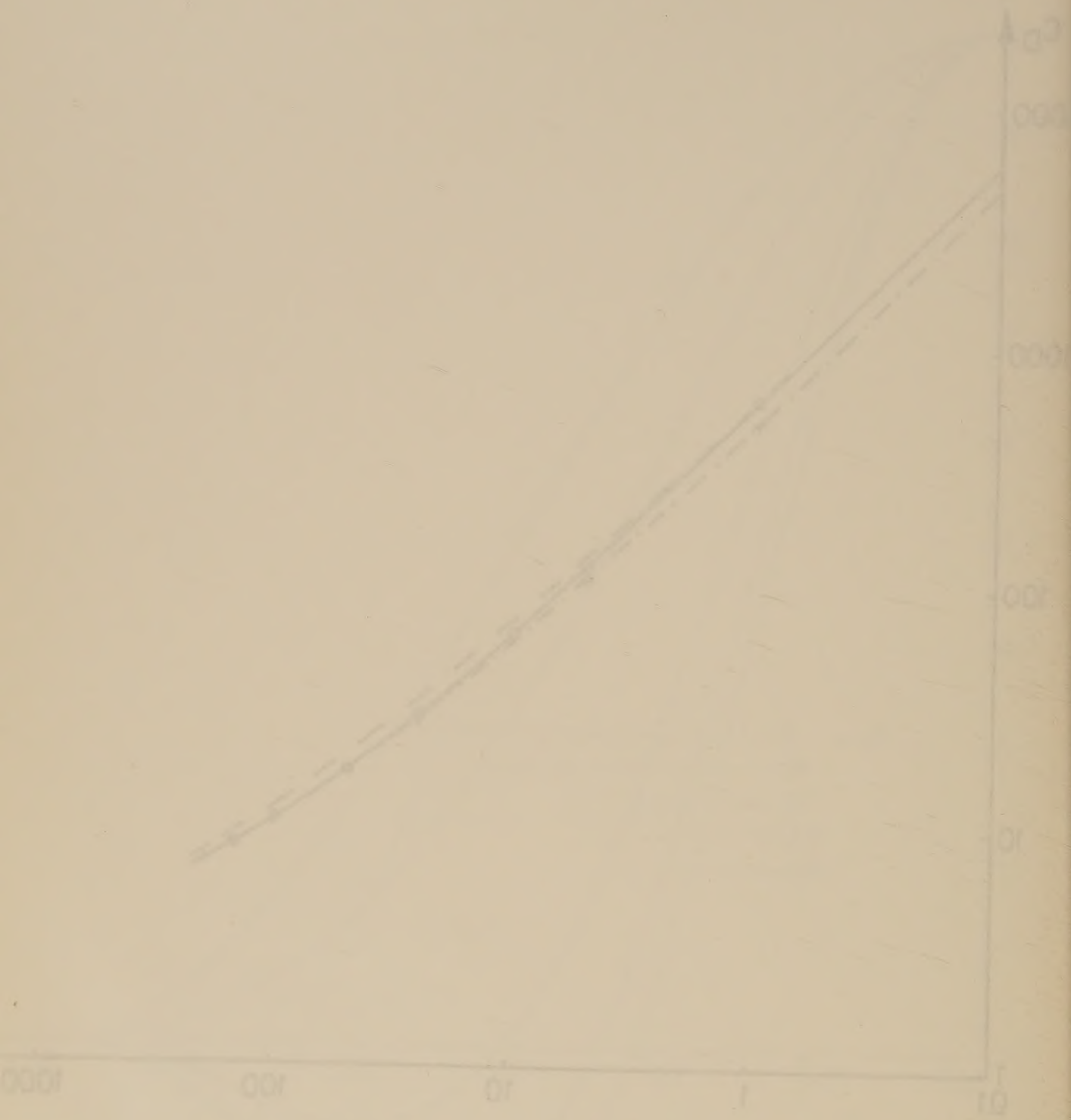


FIGURE 51. The drag coefficient.
 — Taylor and Hargrett (1961)
 - - - Garratt and Wilson (1971)
 — This investigation

